MAT3400/4400 - Spring 2023 - Exercises for Friday, Feb. 10

Extra-exercise 7

Let $(X, \mathcal{A})$ be a measurable space, $g : X \to \mathbb{R}$ be measurable (w.r.t. $\mathcal{A}$) and $c \in \mathbb{R}$. Check that the function $c \cdot g$ is measurable (w.r.t. $\mathcal{A}$).

Extra-exercise 8

Let $(X, \mathcal{A})$ be a measurable space and set

$$\overline{\mathcal{M}} := \{f : X \to \overline{\mathbb{R}} \mid f \text{ is measurable (w.r.t. } \mathcal{A})\}.$$

Let $f, g \in \overline{M}$ and set $D := \{x \in X \mid f(x) \neq g(x)\}$. Show that $D \in \mathcal{A}$.

Note that if μ is a measure on $(X, \mathcal{A})$, then it makes now sense to say (as in Lindstrøm's book) that $f = g$ *almost everywhere* (w.r.t. μ) when $\mu(D) = 0$.

Extra-exercise 9

Let $(X, \mathcal{A})$ be a measurable space and $E \in \mathcal{A}$. Set

$$\mathcal{A}_E = \{A \cap E \,|\, A \in \mathcal{A}\} = \{B \in \mathcal{A} \mid B \subseteq E\} \subseteq \mathcal{A}\,.$$

a) Check that $\mathcal{A}_E$ is a σ-algebra on E, so $(E, \mathcal{A}_E)$ is a measurable space.

Then convince yourself that if μ is a measure on $(X, \mathcal{A})$, then the restriction μ_E of μ to $\mathcal{A}_E$, given by

$$\mu_E(B) = \mu(B) \quad \text{for } B \in \mathcal{A}_E\,,$$

is a measure on $(E, \mathcal{A}_E)$.

Let now $\overline{\mathcal{M}}$ be as in the previous exercise and set

$$\overline{\mathcal{M}}_E := \{h : E \to \overline{\mathbb{R}} \mid h \text{ is } \mathcal{A}_E\text{-measurable}\}\,.$$

b) Let $f \in \overline{\mathcal{M}}$ and let f_E denote the restriction of f to E, i.e.,

$$f_E(x) := f(x) \quad \text{for each } x \in E\,.$$

Show that $f_E \in \overline{\mathcal{M}}_E$.

c) Let $h \in \overline{\mathcal{M}}_E$ and define $\tilde{h} : X \to \overline{\mathbb{R}}$ by

$$\tilde{h}(x) = \begin{cases} h(x) & \text{if } x \in E, \\ \\ 0 & \text{if } x \notin E. \end{cases}$$

Show that $\tilde{h} \in \overline{\mathcal{M}}$.

d) Let μ be a measure on $(X, \mathcal{A})$ and assume that $\mu(E^c) = 0$.

Check that if $f \in \overline{\mathcal{M}}$, then $f = \widetilde{f_E}$ almost everywhere (w.r.t. μ).

- From Lindstrøm's book, section 7.3 : 8, 11, 12, 13
- From Lindstrøm's book, section 7.4 : 3, 4