MAT3400/4400 - Spring 2023 - Exercises for Friday, Feb. 17

- From Lindstrøm's book: Exercise 7.4.5 and Exercises 7.5: 4 a)b)c), 5, 6, 7, 12, 13, 14

Extra exercise 10

Let $(X, \mathcal{A}, \mu)$ be a measure space and let $\rho : X \to [0, \infty]$ be measurable. From Exercise 7.5.5 c), we know that $\nu : \mathcal{A} \to [0, \infty]$ given by

$$\nu(A) = \int_A \rho \, d\mu \quad \text{for all } A \in \mathcal{A}$$

is a measure on $\mathcal{A}$. (One sometimes writes instead that ν is given by $d\nu = \rho \, d\mu$.)

Let $f : X \to [0, \infty]$ be measurable. Show that $\int f \, d\nu = \int f \rho \, d\mu$.

(*Hint.* Check first that this holds when f is any simple nonnegative measurable function.)

Extra exercise 11

Let $(X, \mathcal{A}, \mu)$ be a measure space. Set $\overline{\mathcal{M}}^+ = \{f : X \to \overline{\mathbb{R}}_+ \mid f \text{ is measurable w.r.t. } \mathcal{A}\}$ and let $E \in \mathcal{A}$.

We consider the measure space $(E, \mathcal{A}_E, \mu_E)$, where

$$\mathcal{A}_E = \{A \cap E \,|\, A \in \mathcal{A}\} = \{B \in \mathcal{A} \mid B \subseteq E\} \subseteq \mathcal{A}$$

and $\mu_E : \mathcal{A}_E \to \overline{\mathbb{R}}_+$ denotes the restriction of μ to $\mathcal{A}_E$ (cf. Extra Exercise 9).

Set $\overline{\mathcal{M}}_E^+ = \{h : E \to \overline{\mathbb{R}}_+ \mid h \text{ is measurable w.r.t. } \mathcal{A}_E\}$.

a) Let $f \in \overline{\mathcal{M}}^+$ and $B \in \mathcal{A}_E$. We recall that the restriction f_E of f to E belongs to $\overline{\mathcal{M}}_E^+$. Show that

$$\int_B f \, d\mu = \int_B f_E \, d(\mu_E)\,.$$

b) Let $\mu^E : \mathcal{A} \to \overline{\mathbb{R}}_+$ be the measure defined by $\mu^E(A) = \mu(A \cap E)$ for all $A \in \mathcal{A}$.

Let $f \in \overline{\mathcal{M}}^+$ and $A \in \mathcal{A}$. Show that

$$\int_A f \, d(\mu^E) = \int_{A \cap E} f \, d\mu = \int_{A \cap E} f_E \, d(\mu_E)\,.$$

c) Let $h \in \overline{\mathcal{M}}_E^+$, and let $A \in \mathcal{A}$. We recall that the function $\tilde{h} : X \to \overline{\mathbb{R}}$ defined by

$$\tilde{h}(x) = \begin{cases} h(x) & \text{if } x \in E, \\ 0 & \text{if } x \notin E \end{cases}$$

belongs to $\overline{\mathcal{M}}^+$. Show that

$$\int_A \tilde{h} \, d\mu = \int_{A \cap E} h \, d(\mu_E).$$