Solution – Mandatory assignment MAT3400/4400 Spring 2023

Problem 1. Let $(X, \mathcal{A}, \mu)$ be a measure space, Y be a nonempty set, and $\phi : X \to Y$ be a map. We consider

$$\mathcal{B}_\phi := \{B \subseteq Y \mid \phi^{-1}(B) \in \mathcal{A}\}$$

which is known to be a σ-algebra on Y.

Define $\mu_\phi : \mathcal{B}_\phi \to [0, \infty]$ by

$$\mu_\phi(B) := \mu(\phi^{-1}(B)) \quad \text{for all } B \in \mathcal{B}_\phi.$$

a) • *μ_ϕ is a measure on $(Y, \mathcal{B}_\phi)$:*

We first note that $\mu_\phi(\emptyset) = \mu(\phi^{-1}(\emptyset)) = \mu(\emptyset) = 0$. Next, let $\{B_n\}_{n\in\mathbb{N}}$ be a sequence of disjoint sets in $\mathcal{B}_\phi$, and set $A_n := \phi^{-1}(B_n) \in \mathcal{A}$ for each n. Then the A_n's are disjoint, since

$$A_j \cap A_k = \phi^{-1}(B_j) \cap \phi^{-1}(B_k) = \phi^{-1}(B_j \cap B_k) = \phi^{-1}(\emptyset) = \emptyset$$

whenever $j \neq k$. Hence, using that μ is a measure, we get

$$\begin{aligned}\mu_\phi\Big(\bigcup_{n=1}^\infty B_n\Big) &= \mu\Big(\phi^{-1}\Big(\bigcup_{n=1}^\infty B_n\Big)\Big) = \mu\Big(\bigcup_{n=1}^\infty \phi^{-1}(B_n)\Big) = \mu\Big(\bigcup_{n=1}^\infty A_n\Big)\\ &= \sum_{n=1}^\infty \mu(A_n) = \sum_{n=1}^\infty \mu(\phi^{-1}(B_n)) = \sum_{n=1}^\infty \mu_\phi(B_n)\,,\end{aligned}$$

showing that μ_ϕ is a measure.

b) Let $f : Y \to \overline{\mathbb{R}}$ be a measurable function (w.r.t. $\mathcal{B}_\phi$).

• *The composition $f \circ \phi : X \to \overline{\mathbb{R}}$ is measurable (w.r.t. $\mathcal{A}$):*

Let $t \in \mathbb{R}$ and set $B_t := f^{-1}([-\infty, t))$. Note that $B_t \in \mathcal{B}_\phi$ since f is measurable (w.r.t. $\mathcal{B}_\phi$). This means that $\phi^{-1}(B_t) \in \mathcal{A}$. We therefore get that

$$\begin{aligned}(f \circ \phi)^{-1}\Big([-\infty, t)\Big) &= \{x \in X : (f \circ \phi)(x) < t\} = \{x \in X : f(\phi(x)) < t\}\\ &= \{x \in X : \phi(x) \in B_t\} = \phi^{-1}(B_t) \in \mathcal{A}\,.\end{aligned}$$

Since this holds for every $t \in \mathbb{R}$, we have thus shown that $f \circ \phi : X \to \overline{\mathbb{R}}$ is measurable (w.r.t. $\mathcal{A}$).

c) • *For every $f : Y \to [0,\infty]$ which is measurable (w.r.t. $\mathcal{B}_\phi$), we have*

$$\int_Y f \; d\mu_\phi = \int_X f \circ \phi \; d\mu.$$

We first show that this formula holds whenever $f = \mathbf{1}_B$ with $B \in \mathcal{B}_\phi$. Since

$$\mathbf{1}_{\phi^{-1}(B)}(x) = \begin{cases} 1 & \text{if } \phi(x) \in B \\ 0 & \text{otherwise} \end{cases} \;=\; \mathbf{1}_B(\phi(x)) \;=\; (\mathbf{1}_B \circ \phi)(x)$$

for every $x \in X$, we get that

$$\int_Y \mathbf{1}_B \; d\mu_\phi = \mu_\phi(B) = \mu(\phi^{-1}(B)) = \int_X \mathbf{1}_{\phi^{-1}(B)} \; d\mu = \int_X \mathbf{1}_B \circ \phi \; d\mu,$$

as desired. Next, consider a nonnegative simple function, say $f = \sum_{j=1}^n c_j \mathbf{1}_{B_j}$ with $c_1, \dots, c_n \geq 0$ and $B_1, \dots, B_n \in \mathcal{B}$. By linearity of the integral, and using the first part, we get

$$\int_Y f \; d\mu_\phi = \sum_{j=1}^n c_j \int_Y \mathbf{1}_{B_j} \, d\mu_\phi = \sum_{j=1}^n c_j \int_X \mathbf{1}_{B_j} \circ \phi \; d\mu$$

$$= \int_X \Big(\sum_{j=1}^n c_j \mathbf{1}_{B_j} \Big) \circ \phi \; d\mu = \int_X f \circ \phi \; d\mu \,,$$

which shows that the formula holds for all such f's.

Finally, assume $f : Y \to [0,\infty]$ is measurable (w.r.t. $\mathcal{B}_\phi$). Then using Lebesgue's key result, we can find an increasing sequence $\{f_n\}$ of nonnegative simple functions on X converging pointwise to f. Then $\{f_n \circ \phi\}$ is a sequence of nonnegative measurable functions on Y converging pointwise to $f \circ \phi$. Hence, using the MCT (twice) and what we just showed, we get

$$\int_Y f \; d\mu_\phi = \lim_{n\to\infty} \int_Y f_n \; d\mu_\phi = \lim_{n\to\infty} \int_X f_n \circ \phi \; d\mu = \int_X f \circ \phi \; d\mu \,,$$

as we wanted to show.

d) • $\phi(X) \in \mathcal{B}_\phi$ because $\phi^{-1}(\phi(X)) = \{x \in X : \phi(x) \in \phi(X)\} = X \in \mathcal{A}$.

• *Let $B \in \mathcal{B}_\phi$. Then $\mu_\phi(B) = \mu_\phi\Big(B \cap \phi(X)\Big)$:*

Indeed, since

$$\phi^{-1}(B \cap \phi(X)) = \phi^{-1}(B) \cap \phi^{-1}(\phi(X)) = \phi^{-1}(B) \cap X = \phi^{-1}(B),$$

we have $\mu_\phi\Big(B \cap \phi(X)\Big) = \mu\Big(\phi^{-1}(B \cap \phi(X))\Big) = \mu(\phi^{-1}(B)) = \mu_\phi(B)$.

• *Let $B \in \mathcal{B}_\phi$ and $f : Y \to [0, \infty]$ be measurable (w.r.t. $\mathcal{B}_\phi$). Then we have*

$$\int_B f \, d\mu_\phi = \int_{B \cap \phi(X)} f \, d\mu_\phi = \int_{\phi^{-1}(B)} f \circ \phi \, d\mu. \tag{1}$$

Indeed, using c), we get

$$\begin{aligned} \int_B f \, d\mu_\phi &= \int_X f \, \mathbf{1}_B \, d\mu_\phi = \int_X (f \, \mathbf{1}_B) \circ \phi \, d\mu = \int_X (f \circ \phi) \, (\mathbf{1}_B \circ \phi) \, d\mu \\ &= \int_X (f \circ \phi) \, \mathbf{1}_{\phi^{-1}(B)} \, d\mu = \int_{\phi^{-1}(B)} f \circ \phi \, d\mu \,. \end{aligned}$$

To show the first equality, note that

$$\mu_\phi(\phi(X)^c) = \mu_\phi(\phi(X)^c \cap \phi(X)) = \mu_\phi(\emptyset) = 0 \,.$$

As $B \cap \phi(X)^c \subseteq \phi(X)^c$, this implies that $\mu_\phi(B \cap \phi(X)^c) = 0$. Hence we get

$$\int_B f \, d\mu_\phi = \int_{B \cap \phi(X)} f \, d\mu_\phi + \int_{B \cap \phi(X)^c} f \, d\mu_\phi = \int_{B \cap \phi(X)} f \, d\mu_\phi.$$

e) Let $\mathcal{B}$ be a σ-algebra on Y and assume that the map $\phi : X \to Y$ satisfies the following two conditions:

(i) $\phi^{-1}(B) \in \mathcal{A}$ for every $B \in \mathcal{B}$,

(ii) $\phi(X) \in \mathcal{B}$.

• *We have that $\mathcal{B} \subseteq \mathcal{B}_\phi$:*

Indeed, let $B \in \mathcal{B}$. Then $\phi^{-1}(B) \in \mathcal{A}$ by (i), i.e., $B \in \mathcal{B}_\phi$. This shows the asserted inclusion.

Thus, we may restrict μ_ϕ to $\mathcal{B}$ and get a measure on $(Y, \mathcal{B})$, that we also denote by μ_ϕ.

• *Let $B \in \mathcal{B}$ and $f : Y \to [0, \infty]$ be measurable (w.r.t. $\mathcal{B}$). Then equation* (1) *holds*:

We first note that, since $\mathcal{B} \subseteq \mathcal{B}_\phi$, we have that $B \in \mathcal{B}_\phi$. We also have that f is measurable (w.r.t. $\mathcal{B}_\phi$). Indeed, if $t \in \mathbb{R}$, then $f^{-1}([-\infty, t)) \in \mathcal{B}$, so $f^{-1}([-\infty, t)) \in \mathcal{B}_\phi$. Since $\phi(X) \in \mathcal{B}$, this means that we can apply d) to get that equation (1) holds with f and B.

Problem 2. We consider $X = [0, 2\pi]$ (with its standard metric) and $Y = \mathbb{R}^2$ (with the Euclidean metric). Set $\mathcal{A} := \mathcal{B}_{[0,2\pi]}$ and let μ be the Lebesgue measure on $([0, 2\pi], \mathcal{A})$. Further, let $\phi : [0, 2\pi] \to \mathbb{R}^2$ be the map given by

$$\phi(x) := (\cos x, \sin x), \quad 0 \le x \le 2\pi.$$

a) Set $\mathcal{B} := \mathcal{B}_{\mathbb{R}^2}$.

• *The map ϕ satisfies the two conditions stated in Problem 1e)*:

As the component functions of ϕ are continuous, the map ϕ is continuous, so $\phi^{-1}(U)$ is an open subset of $[0, 2\pi]$ whenever U is an open subset of $\mathbb{R}^2$ (cf. MAT2400). This gives that $\phi^{-1}(U) \in \mathcal{A}$, i.e., $U \in \mathcal{B}_\phi$, for every open $U \subseteq \mathbb{R}^2$ (where $\mathcal{B}_\phi$ is defined as in Problem 1). As $\mathcal{B}_\phi$ is a σ-algebra, this implies that $\mathcal{B}_\phi$ contains the σ-algebra generated by the open subsets of $\mathbb{R}^2$, i.e., we have $\mathcal{B} \subseteq \mathcal{B}_\phi$. This says precisely that $\phi^{-1}(B) \in \mathcal{A}$ for every $B \in \mathcal{B}$, i.e., condition (i) in 1e) holds.

Since $\phi(X) = \{(s,t) : s^2 + t^2 = 1\}$ is obviously a closed subset of $\mathbb{R}^2$, we also have $\phi(X) \in \mathcal{B}$, i.e., condition (ii) in 1e) holds.

We can now push forward μ to a measure μ_ϕ on $(\mathbb{R}^2, \mathcal{B}_{\mathbb{R}^2})$, cf. Problem 1.

b) Set $B := \{(s,t) \in \mathbb{R}^2 : 0 \le s, t \le 1\} \in \mathcal{B}_{\mathbb{R}^2}$.

• *$\mu_\phi(\mathbb{R}^2)$ and $\mu_\phi(B)$ can be computed using the definition of μ_ϕ*:

Indeed, since $\phi^{-1}(\mathbb{R}^2) = [0, 2\pi]$ and $\phi^{-1}(B) = [0, \pi/2] \cup \{2\pi\}$ (make a drawing!), we get that

$$\mu_\phi(\mathbb{R}^2) = \mu(\phi^{-1}(\mathbb{R}^2)) = \mu\Big([0, 2\pi]\Big) = 2\pi \quad \text{and}$$
$$\mu_\phi(B) = \mu(\phi^{-1}(B)) = \mu\Big([0, \pi/2]\Big) + \mu(\{2\pi\}) = \pi/2.$$

• *Set $f(s,t) := s^2 + 1$ for all $(s,t) \in \mathbb{R}^2$. Then $\int_B f \, d\mu_\phi$ can be computed as follows*:

We first note that $f : \mathbb{R}^2 \to \mathbb{R}$ is continuous, so it is Borel measurable, i.e., measurable (w.r.t. $\mathcal{B}$). Using the formula in Problem 1, we get

$$\int_B f \, d\mu_\phi = \int_{[0,\pi/2]\cup\{2\pi\}} f(\cos x, \sin x) \, d\mu(x) = \int_{[0,\pi/2]} (\cos^2 x + 1) \, d\mu(x).$$

Since $x \mapsto \cos^2 x + 1$ is continuous, it is Riemann-integrable on $[0, \pi/2]$, so we get

$$\int_B f \, d\mu_\phi = \int_0^{\pi/2} (\cos^2 x + 1) \, dx = \left[\frac{1}{4} \sin(2x) + \frac{3}{2} x\right]_{x=0}^{x=\pi/2} = \frac{3}{4}\pi \, .$$

Problem 3. Let $\mathcal{B}$ denote the Borel σ-algebra on $\mathbb{R}$ and let μ be the Lebesgue measure on $(\mathbb{R}, \mathcal{B})$.

- *Let $f : \mathbb{R} \to [0, \infty]$ be measurable (w.r.t. $\mathcal{B}$). Then we have*

$$\int_{\mathbb{R}} f \, d\mu = \lim_{n\to\infty} \int_{[-n,n]} f \, d\mu$$

Indeed, set $f_n = f \, \mathbf{1}_{[-n,n]}$ for each $n \in \mathbb{N}$. Then $\{f_n\}$ is readily seen to be an increasing sequence of nonnegative measurable functions (w.r.t. $\mathcal{B}$) converging pointwise to f. So the MCT gives that

$$\int_{\mathbb{R}} f \, d\mu = \lim_{n\to\infty} \int_{\mathbb{R}} f_n \, d\mu = \lim_{n\to\infty} \int_{\mathbb{R}} f \, \mathbf{1}_{[-n,n]} \, d\mu = \lim_{n\to\infty} \int_{[-n,n]} f \, d\mu \, .$$

- *Set $f(x) := e^{-|x|}$ for all $x \in \mathbb{R}$. Then $\int_{\mathbb{R}} f \, d\mu$ can be computed as follows*:

Since f is continuous on $\mathbb{R}$, it is Riemann-integrable on $[-n, n]$ for each $n \in \mathbb{N}$. Thus, using the first part, we get

$$\int_{\mathbb{R}} f \, d\mu = \lim_{n\to\infty} \int_{[-n,n]} e^{-|x|} \, d\mu(x) = \lim_{n\to\infty} \int_{-n}^{n} e^{-|x|} \, dx$$

$$= \lim_{n\to\infty} \Big[-2e^{-x} \Big]_{x=0}^{x=n} = \lim_{n\to\infty} 2(1 - e^{-n}) = 2 \, .$$

Problem 4. Let $\mathcal{B}$ denote the Borel σ-algebra on $(0, \infty)$ and let μ be the Lebesgue measure on $((0, \infty), \mathcal{B})$. For each $n \in \mathbb{N}$, let $f_n : (0, \infty) \to \mathbb{R}$ be the function defined by

$$f_n(x) := \frac{n \sin(\frac{x}{n})}{x(1 + x^2)} \, , \quad x > 0.$$

a) • *Each f_n is integrable (w.r.t. μ)* :

Being continuous on $(0, \infty)$, each f_n is measurable (w.r.t. $\mathcal{B}$). Moreover, since $|\sin(t)| \leq t$ for each $t > 0$, we have that

$$|f_n(x)| \leq \frac{x/n}{x(1 + x^2)} = \frac{1}{n(1 + x^2)} \leq \frac{1}{1 + x^2}$$

for every $n \in \mathbb{N}$ and $x > 0$.

Set $g(x) := (1 + x^2)^{-1}$ for $x \geq 0$. We also denote the Lebesgue measure on $[0, \infty]$ by μ. Being continuous, g is Borel measurable on $[0, \infty]$, and

$$\int_{(0,\infty)} g \, d\mu = \int_{[0,\infty)} g \, d\mu \quad \text{(since } \mu(\{0\}) = 0).$$

Hence the MCT gives (in a similar way as in Problem 3) that

$$\int_{(0,\infty)} g\, d\mu = \int_{[0,\infty)} g\, d\mu = \lim_{n\to\infty} \int_{[0,n]} \frac{1}{1+x^2}\, d\mu(x)$$

$$= \lim_{n\to\infty} \int_0^n \frac{1}{1+x^2}\, dx = \lim_{n\to\infty} \arctan n = \pi/2 \;<\; \infty\,.$$

Thus g is integrable on $(0,\infty)$ (w.r.t. μ). As $|f_n| \le g$ on $(0,\infty)$, we get

$$\int_{(0,\infty)} |f_n|\, d\mu \le \int_{(0,\infty)} g \;\; \mathrm{d}\mu < \infty$$

for every $n \in \mathbb{N}$, showing that each f_n is integrable (w.r.t. μ).

b) • *We have that*

$$\lim_{n\to\infty} \int_{(0,\infty)} f_n \; d\mu = \frac{\pi}{2}\,.$$

Indeed, we have seen above that $|f_n| \le g$ on $(0,\infty)$ for each n, and that g is integrable on $(0,\infty)$ (w.r.t. μ). Further, for every $x > 0$, we have

$$\lim_{n\to\infty} \frac{n\,\sin(x/n)}{x} = \lim_{n\to\infty} \frac{\sin(x/n)}{x/n} = 1,$$

hence

$$\lim_{n\to\infty} f_n(x) = \frac{1}{1+x^2} = g(x)\,.$$

Thus, applying the LDCT, we get that

$$\lim_{n\to\infty} \int_{(0,\infty)} f_n \; d\mu = \int_{(0,\infty)} g \; d\mu = \frac{\pi}{2}\,.$$