23rd February, 2023

COURSE MAT3400/4400

Mandatory assignment

Submission deadline

Thursday 9th of MARS 2023, 14:30 in Canvas (canvas.uio.no).

Instructions

Note that you only have **one attempt** to pass the assignment. This means that it will not be possible to hand in a second attempt.

MAT3400 students can choose between scanning handwritten notes or typing the solution directly on a computer (for instance with LaTeX). MAT4400 students **must** use a typesetting software for mathematics. The assignment must be submitted as a single PDF file. Scanned pages must be clearly legible. The submission must contain your name and the course number.

It is expected that you give a clear presentation with all necessary explanations. All aids, including collaboration, are allowed, but the submission must be written by you and reflect your understanding of the subject. If we doubt that you have understood the content you have handed in, we may request that you give an oral account.

To pass this assignment, it is required that you answer in a satisfactory way at least half of the whole problem set. The ten subproblems all count equally. Only students who pass the assignment will be allowed to take the final examination.

Application for postponed delivery

If you need to apply for a postponement of the submission deadline due to illness or other reasons, you have to contact the Student Administration at the Department of Mathematics (e-mail: studieinfo@math.uio.no) no later than the same day as the deadline.

Complete guidelines about delivery of mandatory assignments:

uio.no/english/studies/admin/compulsory-activities/mn-math-mandatory.html

GOOD LUCK!

Problem 1. Let $(X, \mathcal{A}, \mu)$ be a measure space, Y be a nonempty set and let $\phi : X \to Y$ be a map.

We recall from Exercise 7.1.10 in Lindstrøm's book that

$$\mathcal{B}_\phi := \{B \subseteq Y \mid \phi^{-1}(B) \in \mathcal{A}\}$$

is a σ-algebra on Y.

We can now push forward the measure μ to a measure on $(Y, \mathcal{B}_\phi)$. Indeed, define $\mu_\phi : \mathcal{B}_\phi \to [0, \infty]$ by

$$\mu_\phi(B) := \mu(\phi^{-1}(B)) \quad \text{for all } B \in \mathcal{B}_\phi.$$

a) Show that μ_ϕ is a measure on $(Y, \mathcal{B}_\phi)$.

b) Let $f : Y \to \overline{\mathbb{R}}$ be a measurable function (w.r.t. $\mathcal{B}_\phi$). Check that the composition $f \circ \phi : X \to \overline{\mathbb{R}}$ is measurable (w.r.t. $\mathcal{A}$).

c) Show that

$$\int_Y f \, d\mu_\phi = \int_X f \circ \phi \, d\mu$$

for every $f : Y \to [0, \infty]$ which is measurable (w.r.t. $\mathcal{B}_\phi$).

Hint. Start by showing that this formula holds whenever $f = \mathbf{1}_B$ with $B \in \mathcal{B}_\phi$.

d) Check first that $\phi(X) \in \mathcal{B}_\phi$ and $\mu_\phi(B) = \mu_\phi\big(B \cap \phi(X)\big)$ for every $B \in \mathcal{B}_\phi$. Then show that

$$\int_B f \, d\mu_\phi = \int_{B \cap \phi(X)} f \, d\mu_\phi = \int_{\phi^{-1}(B)} f \circ \phi \, d\mu \tag{1}$$

for every $B \in \mathcal{B}_\phi$ and every $f : Y \to [0, \infty]$ which is measurable (w.r.t. $\mathcal{B}_\phi$).

e) Let $\mathcal{B}$ be a σ-algebra on Y and assume that the map $\phi : X \to Y$ satisfies the following two conditions:

(i) $\phi^{-1}(B) \in \mathcal{A}$ for every $B \in \mathcal{B}$,

(ii) $\phi(X) \in \mathcal{B}$.

Check that $\mathcal{B} \subseteq \mathcal{B}_\phi$. Thus, we may restrict μ_ϕ to $\mathcal{B}$ and get a measure on $(Y, \mathcal{B})$, that we also denote by μ_ϕ. Deduce that equation (1) holds for every $B \in \mathcal{B}$ and every $f : Y \to [0, \infty]$ which is measurable (w.r.t. $\mathcal{B}$).

Problem 2. We recall that if M is a metric space, then $\mathcal{B}_M$ denotes the Borel σ-algebra on M. In this problem, we consider the metric spaces $X = [0, 2\pi]$ (with its standard metric) and $Y = \mathbb{R}^2$ (with the Euclidean metric). The point is to illustrate how Problem 1 can be applied.

Set $\mathcal{A} := \mathcal{B}_{[0,2\pi]}$ and let μ be the Lebesgue measure on $([0, 2\pi], \mathcal{A})$. Further, let $\phi : [0, 2\pi] \to \mathbb{R}^2$ be the map given by

$$\phi(x) := (\cos x, \sin x), \quad 0 \leq x \leq 2\pi.$$

a) Set $\mathcal{B} := \mathcal{B}_{\mathbb{R}^2}$. Explain why the map ϕ satisfies the two conditions stated in Problem 1e).

Note that this means that we can push forward μ to a measure μ_ϕ on $(\mathbb{R}^2, \mathcal{B}_{\mathbb{R}^2})$, cf. Problem 1.

b) Set $B := \{(s, t) \in \mathbb{R}^2 : 0 \leq s, t \leq 1\} \in \mathcal{B}_{\mathbb{R}^2}$.

Compute $\mu_\phi(\mathbb{R}^2)$ and $\mu_\phi(B)$.

Compute also $\int_B f \, d\mu_\phi$, where $f(s, t) := s^2 + 1$ for all $(s, t) \in \mathbb{R}^2$.

Problem 3. Let $\mathcal{B}$ denote the Borel σ-algebra on $\mathbb{R}$ and let μ be the Lebesgue measure on $(\mathbb{R}, \mathcal{B})$.

Show that

$$\int_{\mathbb{R}} f \, d\mu = \lim_{n\to\infty} \int_{[-n,n]} f \, d\mu$$

for every $f : \mathbb{R} \to [0, \infty]$ which is measurable (w.r.t. $\mathcal{B}$). Then use this to compute the integral $\int_{\mathbb{R}} f \, d\mu$ when $f(x) := e^{-|x|}$ for all $x \in \mathbb{R}$.

Problem 4. Let $\mathcal{B}$ denote the Borel σ-algebra on $(0, \infty)$ and let μ be the Lebesgue measure on $((0, \infty), \mathcal{B})$. For each $n \in \mathbb{N}$, let $f_n : (0, \infty) \to \mathbb{R}$ be the function defined by

$$f_n(x) := \frac{n \sin(\frac{x}{n})}{x(1 + x^2)}, \quad x > 0.$$

a) Show that each f_n is integrable (w.r.t. μ).

Hint. You may use that $|\sin(t)| \leq t$ for each $t > 0$.

b) Show that

$$\lim_{n\to\infty} \int_{(0,\infty)} f_n \, d\mu = \frac{\pi}{2}.$$