4.2 Compact operators on Hilb. spaces II

We assume H is a separable space, with an o.n.b. $\{u_j\}_{j\in\mathbb{N}}$, $T\in\mathcal{B}(H)$.

Have seen that $\|T\|_2 := \left(\sum_{j=1}^{\infty}\|T(u_j)\|^2\right)^{1/2}$ does not depend of the choice of o.n.b. for H.

We set $\mathcal{HS}(H) := \{T\in\mathcal{B}(H) : \|T\|_2<\infty\}$

$\uparrow$ the Hilb. Schmidt operators on H.

Prop $\boxed{\begin{array}{l}\mathcal{HS}(H) \text{ is a subspace of } \mathcal{K}(H) \text{ containing } \mathcal{F}(H)\text{, which is closed under the } *\text{-operation.}\\ \text{Moreover, } \|\cdot\|_2 \text{ is a norm on } \mathcal{HS}(H) \text{ s.t. } \|T\|\le\|T\|_2 \text{ for all } T\in\mathcal{HS}(H).\end{array}}$

Proof Have seen that $T\in\mathcal{HS}(H) \Rightarrow T^*\in\mathcal{HS}(H)$.

Let $T, T'\in\mathcal{HS}(H)$. Let then $\xi,\xi'\in\ell^2(\mathbb{N})$ be given by $\xi(j)=\|T(u_j)\|$, $\xi'(j)=\|T'(u_j)\|$ $\forall j\in\mathbb{N}$.

Then $\|\xi\|_2=\|T\|_2$ and $\|\xi'\|_2=\|T'\|_2$, so

$$\|T+T'\|_2^2 = \sum_{j=1}^{\infty}\|(T+T')(u_j)\|^2 = \sum_{j=1}^{\infty}\|T(u_j)+T'(u_j)\|^2$$

$$\overset{\Delta\text{-ineq. in } H}{\le} \sum_{j=1}^{\infty}\Big(\underbrace{\underbrace{\|T(u_j)\|}_{\xi(j)}+\underbrace{\|T'(u_j)\|}_{\xi'(j)}}_{(\xi+\xi')(j)}\Big)^2$$

$$= \|\xi+\xi'\|_2^2$$

$$\overset{\Delta\text{-ineq. in } \ell^2(\mathbb{N})}{\le} \left(\|\xi\|_2+\|\xi'\|_2\right)^2$$

$$= \left(\|T\|_2+\|T'\|_2\right)^2 < \infty, \text{ so}$$

$\underline{T+T'\in\mathcal{HS}(H)}$ and $\underline{\|T+T'\|_2\le\|T\|_2+\|T'\|_2}$.

- If $\lambda\in\mathbb{F}$, then it is easy to see that $\lambda T\in\mathcal{HS}(H)$ and $\|\lambda T\|_2=|\lambda|\,\|T\|_2$.
- Assume $T\in\mathcal{HS}(H)$, $\|T\|_2=0$. Then $\sum_{j=1}^{\infty}\|T(u_j)\|^2=0$, so $\|T(u_j)\|=0$ $\forall j\in\mathbb{N}$, hence $T(u_j)=0$ $\forall j\in\mathbb{N}$, which implies that $\underline{T=0}$.

So we have shown $\mathcal{HS}(H)$ is a subspace of $\mathcal{B}(H)$ and that $\|\cdot\|_2$ is a norm on it.

- $\boxed{\|T\|\le\|T\|_2}$: Let $x\in H\setminus\{0\}$, and set $v_1 := \frac{1}{\|x\|}x$.
$\underline{T\in\mathcal{HS}(H)}$ We can now pick an o.n.b. $\{v_j\}_{j\ge2}$ for $M:=\{x\}^\perp$, so $\{v_j\}_{j\in\mathbb{N}}$ is an o.n.b. for H.

Then $\|T(x)\|^2 = \|T(\|x\|v_1)\|^2 = \|x\|^2\|T(v_1)\|^2$
$\le \|x\|^2\left(\sum_{j=1}^{\infty}\|T(v_j)\|^2\right) = \|x\|^2\,\|T\|_2^2$

So $\underline{\|T(x)\|\le\|T\|_2\|x\|}$. Hence $\underline{\|T\|\le\|T\|_2}$.

- $\underline{\text{Assume } T\in\mathcal{HS}(H)\text{. Then } T\in\mathcal{K}(H)}$:

Let P_n be the orth. proj. of H on $\mathrm{Span}\{u_1,\dots,u_n\}$ for each $n\in\mathbb{N}$.

Set $T_n := TP_n\in\mathcal{F}(H)$ for each n.

Then $\|T-T_n\|\le\|T-T_n\|_2 = \left(\sum_{j=1}^{\infty}\|T(u_j)-TP_n(u_j)\|^2\right)^{1/2}$

$= \sum_{j=n+1}^{\infty}\|T(u_j)\|^2 \to 0$ as $n\to\infty$

(since $\sum_{j=1}^{\infty}\|T(u_j)\|^2<\infty$)

So $T=\lim_{n\to\infty}\underbrace{T_n}_{\in\mathcal{F}(H)}$ (in op. norm)

Hence $T\in\overline{\mathcal{F}(H)}^{\|\cdot\|}=\mathcal{K}(H)$.

We leave to prove that $\mathcal{F}(H)\subseteq\mathcal{HS}(H)$ as an exercise.

Example $H = L^2([a,b], \mathcal{A}, \mu)$ ← Leb. measure.

$\mathcal{A}$: Leb. measurable subsets of $[a,b]$

$K : [a,b] \times [a,b] \to \mathbb{C}$ continuous.

T_K = the associated integral operator on H, (so T_K is determined on $C([a,b])$ by $[T_K(f)](s) = \int_a^b K(s,t) f(t)\, dt$)

Then $T_K \in \mathcal{HS}(H) \subseteq \mathcal{K}(H)$:

Let $\mathcal{B} = \{[u_j]\}_{j \in \mathbb{N}}$ be the o.n.b. for H which one obtains by applying the Gram-Schmidt process on the set $\{\underset{v_1}{1}, \underset{v_2}{t}, \underset{v_3}{t^2}, \underset{v_4}{t^3}, \dots\}$.

Note that each u_j is then continuous and real-valued, so $\overline{u_j} = u_j$ $\forall j$.

$\left(u_1 := \frac{1}{\|v_1\|} v_1 \;;\quad w_2 = v_2 - \underbrace{\langle v_2, u_1 \rangle}_{\in \mathbb{R}} u_1 \,,\; u_2 := \frac{1}{\|w_2\|} w_2 \;;\; \cdots \right)$

Let $s \in [a,b]$. Let $k_s \in C([a,b])$ be defined by $k_s(t) = K(s,t)$, $t \in [a,b]$.

Note that
$$[T_K(u_j)](s) = \int_a^b K(s,t) u_j(t)\, dt = \int_a^b k_s(t) \overline{u_j(t)}\, dt = \langle [k_s], [u_j] \rangle$$

Since $\|[k_s]\|_2^2 = \sum_{j=1}^{\infty} |\langle [k_s], [u_j] \rangle|^2$ (Parseval's identity)

we get that
$$\sum_{j=1}^{\infty} |[T_K(u_j)](s)|^2 = \sum_{j=1}^{\infty} |\langle [k_s], [u_j] \rangle|^2 = \|[k_s]\|_2^2$$

Thus,
$$\begin{aligned}
\sum_{j=1}^{\infty} \|T_K([u_j])\|_2^2 &= \sum_{j=1}^{\infty} \left(\int_a^b |[T_K(u_j)](s)|^2 \, d\mu(s) \right) \\
&\overset{\text{MCT}}{=} \int_a^b \left(\sum_{j=1}^{\infty} |[T_K(u_j)](s)|^2 \right) d\mu(s) \\
&= \int_a^b \|[k_s]\|_2^2 \, d\mu(s) \\
&= \int_a^b \left(\int_a^b \underbrace{|k_s(t)|^2}_{|K(s,t)|^2} d\mu(t) \right) d\mu(s) \\
&= \int_a^b \int_a^b |K(s,t)|^2 \, dt\, ds \quad \left(\text{since } K \text{ is cont.}\right) \\
&< \infty .
\end{aligned}$$

Hence $T_K \in \mathcal{HS}(H)$, and $\|T_K\|_2 = \int_a^b \int_a^b |K(s,t)|^2 \, dt\, ds$

4.3 The spectral theorem for compact self-adj. operators

$H \neq \{0\}$ Hilbert space (over $\mathbb{F}$).

Let $T \in \mathcal{B}(H)$, $\lambda \in \mathbb{F}$. Set $E_\lambda^T := \{x \in H : T(x) = \lambda x\}$

Then λ is an eigenvalue of T $\Leftrightarrow$ $E_\lambda^T \neq \{0\}$, in which case non-zero vectors in E_λ^T are called eigenvectors of T ass. with λ.

In general, T may have no eigenvalues.

Let us say that T is diagonalizable if there exists an o.n.b. for H which consists of eigenvectors of T.

The spectral theorem for comp. self-adj. op.:

Assume $T \in \mathcal{K}(H)$ is self-adjoint. Then T is diagonalizable

More precisely, assume $T \neq 0$ (and write E_λ instead of E_λ^T). Then

a) $L := \{\lambda \in \mathbb{F} : \lambda$ is a nonzero eigenvalue of $T\}$ is a non-empty countable subset of $[-\|T\|, \|T\|]$ which always contains$\underbrace{\ }_{\text{at least}}$ $-\|T\|$ or $\|T\|$.

b) If L is countably infinite and $\{\lambda_j, j \in \mathbb{N}\}$ is a listing of its elements, then $\lim_{j\to\infty} \lambda_j = 0$.

c) $\dim(E_\lambda) < \infty$ for all $\lambda \in L$.

d) For each $\lambda \in L$, let $\mathcal{E}_\lambda$ be an o.n.b for E_λ and set $\mathcal{E}' := \bigcup_{\lambda \in L} \mathcal{E}_\lambda$. Then $\mathcal{E}'$ is an o.n.b. for $\overline{T(H)} = \ker(T)^\perp$

e) If $\underbrace{\ker(T)}_{E_0} = \{0\}$ (i.e. 0 is not an eigenvalue of T), then $\mathcal{E}_0 := \emptyset$. Otherwise, let $\mathcal{E}_0$ be an o.n.b. for $\ker(T)$.

Set $\mathcal{E} := \mathcal{E}_0 \cup \mathcal{E}'$. Then $\mathcal{E}$ is an o.n.b. for H which consists of eigenvectors for T.

f) For each $\lambda \in L$, set $P_\lambda :=$ the orth. proj. of H on E_λ.

Then $\boxed{T = \sum_{\lambda \in L} \lambda P_\lambda \quad \text{(w.r.t. operator norm)}}$,

the spectral decomposition of T

(so, if L is countably infinite, we have

$$\|T - \sum_{j=1}^{n} \lambda_j P_{\lambda_j}\| \to 0 \text{ as } n\to\infty$$

where $\{\lambda_j\}$ is as in b))

We will need some lemmas.

Lemma Assume $T \in \mathcal{B}(H)$ is self-adj. and λ is an eigenvalue of T. Then $\lambda \in \mathbb{R}$.
Moreover, if λ' is also an eigenvalue of T and $\lambda' \neq \lambda$, then $E_\lambda \perp E_{\lambda'}$

Proof Let $x \in E_\lambda$, $\|x\| = 1$. Then

$$\lambda = \lambda \|x\|^2 = \lambda \langle x, x \rangle = \langle \underbrace{\lambda x}_{= T(x)}, x \rangle \in W_T \subseteq \mathbb{R}$$

If $y \in E_{\lambda'}$, then

$$\lambda \langle x, y \rangle = \langle T(x), y \rangle = \langle x, \underbrace{T(y)}_{= \lambda' y} \rangle = \lambda' \langle x, y \rangle$$

so $\langle x, y \rangle = 0$ (since $\lambda' \neq \lambda$).