

Carathéodory ~~extension~~ theorem

①

Assume X is a set, $S \subset \mathcal{P}(X)$, $\emptyset \in S$,
and $\mu: S \rightarrow [0, +\infty]$ is a map s.t. $\mu(\emptyset) = 0$.
Can we extend μ to a measure on
 $(X, \sigma(S))$?

Define $\mu^*: \mathcal{P}(X) \rightarrow [0, +\infty]$ by

$$\mu^*(A) = \inf \left\{ \sum_{n=1}^{\infty} \mu(A_n) : A_n \in S, A \subset \bigcup_{n=1}^{\infty} A_n \right\}$$

In general μ^* is neither a measure
nor it extends μ . But we have:

Lemma

The map μ^* has the following properties:

- 1) $\mu^*(\emptyset) = 0$;
- 2) if $A \subset B$, then $\mu^*(A) \leq \mu^*(B)$;
- 3) $\mu^*\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} \mu^*(A_n)$.

Proof

1) Taking $A_n = \emptyset$, we get $\mu^*(\emptyset) \leq \sum_{n=1}^{\infty} \mu(\emptyset) = 0$,
so $\mu^*(\emptyset) = 0$.

2) If $A \subset B$ and $B \subset \bigcup_{n=1}^{\infty} A_n$ for some $A_n \in S$,
then $A \subset \bigcup_{n=1}^{\infty} A_n$. Hence $\mu^*(A) \leq \mu^*(B)$.

3) If $\mu^*(A_n) = +\infty$ for some n , there is
nothing to prove. Therefore assume
that $\mu^*(A_n) < \infty$ for all n .

Fix $\varepsilon > 0$. Then we can find sets

$A_{nk} \in \mathcal{S}$ s.t. $A_n \subset \bigcup_{k=1}^{\infty} A_{nk}$ and

$$\sum_{k=1}^{\infty} \mu(A_{nk}) < \mu^*(A_n) + \frac{\varepsilon}{2^n}.$$

Then $\bigcup_{n=1}^{\infty} A_n \subset \bigcup_{n=1}^{\infty} \bigcup_{k=1}^{\infty} A_{nk}$, so

$$\begin{aligned} \mu^*\left(\bigcup_{n=1}^{\infty} A_n\right) &\leq \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \mu(A_{nk}) \\ &< \sum_{n=1}^{\infty} \left(\mu^*(A_n) + \frac{\varepsilon}{2^n}\right) \\ &= \sum_{n=1}^{\infty} \mu^*(A_n) + \varepsilon. \end{aligned}$$

As $\varepsilon > 0$ was arbitrary, we conclude that $\mu^*\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} \mu^*(A_n)$. $\square$

Any map $\mu^*: \mathcal{P}(X) \rightarrow \{0, +\infty\}$ with properties 1)-3) is called an outer measure on X .

It turns out, every outer measure defines a measure on a certain σ -algebra.

Theorem (Carathéodory)

Assume X is a set and $\mu^*: \mathcal{P}(X) \rightarrow \{0, +\infty\}$ is an outer measure. Let $\mathcal{B} \subset \mathcal{P}(X)$ be the collection of sets A s.t.

$$\mu^*(B) = \mu^*(A \cap B) + \mu^*(A^c \cap B) \quad \forall B \subset X$$

(such sets A are called Carathéodory measurable with respect to μ^*).

Then $\mathcal{B}$ is a σ -algebra and $\mu \stackrel{\text{def}}{=} \mu^*|_{\mathcal{B}}$ is a measure on $(X, \mathcal{B})$. ②

Proof

Obviously, $\emptyset \in \mathcal{B}$, and if $A \in \mathcal{B}$, then $A^c \in \mathcal{B}$.
Let us show that $\mathcal{B}$ is also closed under finite unions (so $\mathcal{B}$ is an algebra).

Assume $A_1, A_2 \in \mathcal{B}$, and take $B \subset X$.

Note that for any $A \subset X$ we have

$$\begin{aligned}\mu^*(B) &= \mu^*((A \cap B) \cup (A^c \cap B)) \\ &\leq \mu^*(A \cap B) + \mu^*(A^c \cap B).\end{aligned}$$

In particular,

$$\mu^*(B) \leq \mu^*((A_1 \cup A_2) \cap B) + \mu^*((A_1 \cup A_2)^c \cap B).$$

On the other hand,

$$\begin{aligned}\mu^*(B) &= \mu^*(A_1 \cap B) + \mu^*(A_1^c \cap B) \\ &= \mu^*(A_1 \cap A_2 \cap B) + \mu^*(A_1 \cap A_2^c \cap B) \\ &\quad + \mu^*(A_1^c \cap A_2 \cap B) + \mu^*(A_1^c \cap A_2^c \cap B) \\ &\geq \mu^*((A_1 \cup A_2) \cap B) + \mu^*((A_1 \cup A_2)^c \cap B),\end{aligned}$$

since $(A_1 \cup A_2)^c \subset A_1^c \cap A_2^c$ and

$$A_1 \cup A_2 = (A_1 \cap A_2) \cup (A_1 \cap A_2^c) \cup (A_1^c \cap A_2).$$

hence

$$\mu^*(B) = \mu^*((A_1 \cup A_2) \cap B) + \mu^*((A_1 \cup A_2)^c \cap B),$$

so $A_1 \cup A_2 \in \mathcal{B}$.

Finally, let us show that $\mathcal{B}$ is closed under countable unions.

Take $A_n \in \mathcal{B}$. Replacing A_n by

$$A_n \setminus \bigcup_{k=1}^{n-1} A_k = A_n \cap \left(\bigcup_{k=1}^{n-1} A_k \right)^c \in \mathcal{B},$$

we may assume that A_n are disjoint.

Let $A = \bigcup_{n=1}^{\infty} A_n$, $\tilde{A}_n = \bigcup_{k=1}^n A_k$. Take $B \subset X$.

Note that we have

$$\begin{aligned} \mu^*(\tilde{A}_n \cap B) &= \mu^*(A_n \cap \tilde{A}_n \cap B) + \mu^*(A_n^c \cap \tilde{A}_n \cap B) \\ &= \mu^*(A_n \cap B) + \mu^*(\tilde{A}_{n-1} \cap B), \end{aligned}$$

as $A_n \cap \tilde{A}_n = A_n$ and $A_n^c \cap \tilde{A}_n = \tilde{A}_{n-1}$.

Doing the same for $\mu^*(\tilde{A}_{n-1} \cap B)$ and so on, we get

$$\mu^*(\tilde{A}_n \cap B) = \sum_{k=1}^n \mu^*(A_k \cap B).$$

Then

$$\begin{aligned} \mu^*(B) &= \mu^*(\tilde{A}_n \cap B) + \mu^*(\tilde{A}_n^c \cap B) \\ &\quad (\text{as } \tilde{A}_n \in \mathcal{B}) \\ &= \sum_{k=1}^n \mu^*(A_k \cap B) + \mu^*(\tilde{A}_n^c \cap B) \\ &\geq \sum_{k=1}^n \mu^*(A_k \cap B) + \mu^*(A^c \cap B) \\ &\quad (\text{as } A^c \subset \tilde{A}_n^c) \end{aligned}$$

Letting $n \rightarrow \infty$ we get

$$\begin{aligned} \mu^*(B) &\geq \sum_{k=1}^{\infty} \mu^*(A_k \cap B) + \mu^*(A^c \cap B) \\ &\geq \mu^*(A \cap B) + \mu^*(A^c \cap B) \\ &\quad (\text{as } A \cap B = \bigcup_{k=1}^{\infty} (A_k \cap B)) \end{aligned}$$

Since the opposite inequality always holds, we conclude that $A \in \mathcal{B}$.
From the proof we also get that

$$\mu^*(B) = \sum_{k=1}^{\infty} \mu^*(A_k \cap B) + \mu^*(A^c \cap B).$$

Taking $B = A$ we obtain

$$\mu^*\left(\bigcup_{k=1}^{\infty} A_k\right) = \sum_{k=1}^{\infty} \mu^*(A_k).$$

Therefore $\mu^*|_{\mathcal{B}}$ is a measure. $\square$

