

Measurable maps

(1)

Proposition

Assume $(X, \mathcal{B})$ is a measurable space, (Y, d) is a metric space. Assume $(f_n: (X, \mathcal{B}) \rightarrow Y)_{n=1}^{\infty}$ is a sequence of measurable maps s.t. $(f_n(x))_{n=1}^{\infty}$ is convergent $\forall x \in X$. Then the map $f: X \rightarrow Y$, $f(x) = \lim_{n \rightarrow \infty} f_n(x)$, is measurable (with respect to $\mathcal{B}$ and $\mathcal{B}(Y)$).

Proof

It suffices to show that if $F \subset Y$ is a closed subset, then $f^{-1}(F) \in \mathcal{B}$. But this follows from the equality

$$(*) \quad f^{-1}(F) = \bigcap_{k=1}^{\infty} \bigcup_{m=1}^{\infty} \bigcap_{n=m}^{\infty} f_n^{-1}(\{y: d(y, F) \leq \frac{1}{k}\})$$

where $d(y, F) = \inf\{d(y, z): z \in F\}$.

Note that the function $y \mapsto d(y, F)$ is continuous, since

$$|d(y, F) - d(z, F)| \leq d(y, z),$$

hence $\{y: d(y, F) \leq \frac{1}{k}\}$ is closed.

To prove $(*)$, assume first that $x \in f^{-1}(F)$.

Let m be s.t. $d(f(x), f_n(x)) \leq \frac{1}{k} \quad \forall n \geq m$.

Then $d(f_n(x), F) \leq \frac{1}{k}$, so

$$x \in \bigcap_{n=m}^{\infty} f_n^{-1}(\{y: d(y, F) \leq \frac{1}{k}\}).$$

this proves that $C \subset (x)$.

For the opposite inclusion, take
 $x \in \bigcap_{k=1}^{\infty} \bigcup_{n=1}^{\infty} \bigcap_{m=n}^{\infty} f_m^{-1}(\{y: d(y, F) \leq \frac{1}{k}\}).$

Fix k . Then $x \in \bigcap_{n=m}^{\infty} f_n^{-1}(\{y: d(y, F) \leq \frac{1}{k}\})$
 for some m , that is,

$$d(f_n(x), F) \leq \frac{1}{k} \quad \forall n \geq m.$$

On the other hand, $f_n(x) \rightarrow f(x)$, so there
 is m' s.t. $d(f_n(x), f(x)) \leq \frac{1}{k} \quad \forall n \geq m'$.

Then, for all $n \geq \max\{m, m'\}$, we have

$$d(f(x), F) \leq d(f(x), f_n(x)) + d(f_n(x), F) \leq \frac{2}{k},$$

so there is $y_n \in F$ s.t.

$$d(f(x), y_n) \leq \frac{3}{k}.$$

Then $f(x) = \lim_{k \rightarrow \infty} y_k \in F$, as F is closed. □

Consider a measurable space $(X, \mathcal{B})$.

Properties of measurable functions:

- 1) if $f, g: (X, \mathcal{B}) \rightarrow \mathbb{C}$ are measurable,
 then $f+g, f \cdot g$ ($(fg)(x) = f(x)g(x)$)
 and cf ($c \in \mathbb{C}$) are measurable;
- 2) if $f: (X, \mathcal{B}) \rightarrow \mathbb{C}$ is measurable
 and $h: \mathbb{C} \rightarrow \mathbb{C}$ is Borel, then
 $h \circ f$ is measurable;

3) if $X = \bigcup_{n=1}^{\infty} A_n$, $A_n \in \mathcal{B}$, and ②

$f|_{A_n} : (A_n, \mathcal{B}_{A_n}) \rightarrow \mathbb{C}$ are measurable for all n , then f is measurable;

4) if $f(x) = \lim_{n \rightarrow \infty} f_n(x) \quad \forall x \in X$ and f_n are measurable for all n , then f is measurable.

Proof

2)-4) follows from similar general statements for measurable maps.

To prove 1), consider the map

$$F: X \rightarrow \mathbb{C}^2, \quad F(x) = (f(x), g(x)).$$

we claim that F is measurable. Indeed, it suffices to show that $F^{-1}(A \times B) \in \mathcal{B}$

$\forall$ Borel sets $A, B \subset \mathbb{C}$, since such $A \times B$ generate $\mathcal{B}(\mathbb{C}^2)$ as a σ -algebra (it suffices to take parallelepipeds). But

$$\begin{aligned} F^{-1}(A \times B) &= \{x : (f(x), g(x)) \in A \times B\} \\ &= f^{-1}(A) \cap g^{-1}(B) \in \mathcal{B}, \end{aligned}$$

so F is measurable.

Then $f+g$ is the composition

$$X \xrightarrow{F} \mathbb{C}^2 \xrightarrow{\quad} \mathbb{C}$$

$$(z_1, z_2) \mapsto z_1 + z_2$$

which is measurable, since the last

map is continuous, hence Borel.

Similarly f, g and cf are measurable
as compositions

$$\mathbb{Q} \times \xrightarrow{F} \mathbb{C}^2 \rightarrow \mathbb{C}$$
$$(z_1, z_2) \mapsto z_1 z_2$$

$$\text{and } X \xrightarrow{f} \mathbb{C} \xrightarrow{\times c} \mathbb{C}$$

□