

Completions of measure spaces

Def

A measure space $(X, \mathcal{B}, \mu)$ is called complete if whenever $A \in \mathcal{B}$, $\mu(A) = 0$, we have $B \in \mathcal{B}$ for all $B \subset A$.

Any measure space $(X, \mathcal{B}, \mu)$ can be completed as follows:

Let $\overline{\mathcal{B}}$ be the σ -algebra generated by $\mathcal{B}$ and all sets $B \subset X$ s.t. there exists $A \in \mathcal{B}$ with $B \subset A$ and $\mu(A) = 0$.

Proposition

The σ -algebra $\overline{\mathcal{B}}$ can also be described as follows:

$$\overline{\mathcal{B}} = \{ B \subset X : A_1 \subset B \subset A_2 \text{ for some } A_1, A_2 \in \mathcal{B} \text{ with } \mu(A_2 \setminus A_1) = 0 \}$$

With B, A_1, A_2 as above, we define

$$\overline{\mu}(B) \stackrel{\text{def}}{=} \mu(A_1) = \mu(A_2).$$

Then $(X, \overline{\mathcal{B}}, \overline{\mu})$ is a complete measure space.

Proof

Consider the collection of sets

$$\tilde{\mathcal{B}} = \{ B \subset X : A_1 \subset B \subset A_2 \text{ for some } A_1, A_2 \in \mathcal{B} \text{ with } \mu(A_2 \setminus A_1) = 0 \}.$$

This is a σ -algebra: $\emptyset \in \tilde{\mathcal{B}}$ (take $A_1 = A_2 = \emptyset$);

if $B \in \tilde{\mathcal{B}}$ and A_1, A_2 are as above, then

$A_2 \subset B \subset A_1$ and $\mu(A_1 \setminus A_2) = \mu(A_2 \setminus A_1) = 0$,
 so $B \in \tilde{\mathcal{B}}$. Finally, assume $(B_n)_{n=1}^{\infty} \subset \tilde{\mathcal{B}}$.

Let $A'_n, A''_n \in \mathcal{B}$ be s.t.

$$A'_n \subset B_n \subset A''_n, \quad \mu(A''_n \setminus A'_n) = 0.$$

Put $A_1 = \bigcup_{n=1}^{\infty} A'_n$, $A_2 = \bigcup_{n=1}^{\infty} A''_n$. Then

$$A_1 \subset \bigcup_{n=1}^{\infty} B_n \subset A_2, \quad A_2 \setminus A_1 \subset \bigcup_{n=1}^{\infty} (A''_n \setminus A'_n),$$

so $\mu(A_2 \setminus A_1) \leq \sum_{n=1}^{\infty} \mu(A''_n \setminus A'_n) = 0$, hence $\bigcup_{n=1}^{\infty} B_n \in \tilde{\mathcal{B}}$.

Thus, $\tilde{\mathcal{B}}$ is a σ -algebra. It contains $\mathcal{B}$
 and all sets $B \subset X$ s.t. $B \subset A$ and $\mu(A) = 0$
 for some $A \in \mathcal{B}$. Hence $\overline{\mathcal{B}} \subset \tilde{\mathcal{B}}$. On the

other hand, if $B \in \tilde{\mathcal{B}}$, $A_1 \subset B \subset A_2$,

$A_1, A_2 \in \mathcal{B}$ and $\mu(A_2 \setminus A_1) = 0$, then

$$B = A_1 \cup (B \setminus A_1) \text{ and } B \setminus A_1 \subset A_2 \setminus A_1.$$

Hence $B \in \overline{\mathcal{B}}$. Thus $\overline{\mathcal{B}} = \tilde{\mathcal{B}}$.

Next, we need to show that $\bar{\mu}$ is

well-defined. Assume

$$A_1 \subset B \subset A_2, \quad A'_1 \subset B \subset A'_2,$$

with $A_1, A_2, A'_1, A'_2 \in \mathcal{B}$, $\mu(A_2 \setminus A_1) = \mu(A'_2 \setminus A'_1) = 0$.

Then $A_1 \setminus A'_1 \subset A'_2 \setminus A'_1$, hence $\mu(A_1 \setminus A'_1) = 0$.

It follows that

$$\mu(A_1) = \mu(A_1 \cap A'_1).$$

For the same reason $\mu(A'_1) = \mu(A_1 \cap A'_1)$.

Therefore $\mu(A_1) = \mu(A'_1)$, so $\bar{\mu}$ is well-defined.

Finally, we have to check that μ is a measure. Assume $(B_n)_{n=1}^{\infty}$ is a sequence of disjoint sets in $\overline{B}$. As above, choose $A_n', A_n'' \in B$ s.t.

$$A_n' \subset B_n \subset A_n'' \text{ and } \mu(A_n'' \setminus A_n') = 0.$$

Then for $A_1 = \bigcup_{n=1}^{\infty} A_n'$ and $A_2 = \bigcup_{n=1}^{\infty} A_n''$ we have $A_1 \subset \bigcup_{n=1}^{\infty} B_n \subset A_2$ and $\mu(A_2 \setminus A_1) = 0$.

hence

$$\begin{aligned} \mu\left(\bigcup_{n=1}^{\infty} B_n\right) &= \mu(A_1) = \mu\left(\bigcup_{n=1}^{\infty} A_n'\right) = \sum_{n=1}^{\infty} \mu(A_n') \\ &= \sum_{n=1}^{\infty} \mu(B_n). \end{aligned}$$

We also have $\mu(\emptyset) = \mu(\emptyset) = 0$. □

Def

If μ is a Borel measure on a metric space (X, d) , then the completion $\overline{B}(X)$ of the Borel σ -algebra with respect to μ is called the σ -algebra of μ -measurable sets.

For $\mu = \lambda_n$ on $\mathbb{R}^n$ we talk about the σ -algebra of Lebesgue measurable sets.

Instead of $\overline{\lambda}_n$ we still write λ_n and call it the Lebesgue measure. A function

$f: \mathbb{R}^n \rightarrow \mathbb{C}$ measurable with respect to the σ -algebra of Lebesgue measurable sets is called Lebesgue measurable.

the following result shows that any Lebesgue measurable function coincides with a Borel function a.e.

Proposition

Assume $(X, \mathcal{B}, \mu)$ is a measure space and consider its completion $(X, \overline{\mathcal{B}}, \overline{\mu})$. Assume $f: X \rightarrow \mathbb{C}$ is $\overline{\mathcal{B}}$ -measurable. Then there is a $\mathcal{B}$ -measurable function $g: X \rightarrow \mathbb{C}$ s.t. $f = g$ $\overline{\mu}$ -a.e.

Proof
By considering separately $(\operatorname{Re} f)_+$, $(\operatorname{Im} f)_+$ we may assume that $f \geq 0$.

Assume first that f is simple,

$$f = \sum_{k=1}^n c_k \pi_{B_k}, \quad c_k \geq 0, \quad B_k \in \overline{\mathcal{B}}.$$

Let $A_k \in \mathcal{B}$ be s.t. $A_k \subset B_k$, $\overline{\mu}(B_k \setminus A_k) = 0$.
Then define $g = \sum_{k=1}^n c_k \pi_{A_k}$. We have

$0 \leq g \leq f$ and $f = g$ $\overline{\mu}$ -a.e., namely,

$$\{x: f(x) \neq g(x)\} \subset \bigcup_{k=1}^n (B_k \setminus A_k).$$

For general $f \geq 0$ choose simple

$\mathcal{B}$ -measurable functions f_n s.t.
 $0 \leq f_1 \leq f_2 \leq \dots$, $f_n \uparrow f$ pointwise.

Then we can find simple $\mathcal{B}$ -measurable functions g_n s.t. $0 \leq g_n \leq f_n$ and the set $A_n = \{x: f_n(x) \neq g_n(x)\}$ has measure zero.

Consider $\tilde{g}_n = \max \{g_1, \dots, g_n\}$.

③

Then

$$0 \leq \tilde{g}_1 \leq \tilde{g}_2 \leq \dots, \quad \tilde{g}_n \leq f_n \quad \text{and}$$

$$\{x : \tilde{g}_n(x) < f_n(x)\} \subset A_n.$$

Define $g(x) = \lim_{n \rightarrow \infty} \tilde{g}_n(x)$. Then g is

$\mathbb{B}$ -measurable, $g \leq f$ and

$$\{x : g(x) < f(x)\} \subset \bigcup_{n=1}^{\infty} A_n, \text{ so } g = f \text{ } \mu\text{-a.e.}$$

