

Regularity of Borel measures

Def Assume X is a metric space and μ is a Borel measure on X . Then μ is called outer regular, if

$$\mu(A) = \inf_{\substack{A \subset \bigcup U \\ U \text{ is open}}} \mu(U) \quad \forall \text{ Borel } A \subset X,$$

and it is called inner regular, if

$$\mu(A) = \sup_{\substack{K \subset A \\ K \text{ is compact}}} \mu(K) \quad \forall \text{ Borel } A \subset X,$$

and μ is called regular if it is both outer and inner regular.

Warning

Sometimes by inner regularity it is meant that $\mu(A) = \sup_{\substack{F \subset A \\ F \text{ closed}}} \mu(F)$, while our definition of regularity is called tightness.

Theorem

Assume X is a metric space and μ is a Borel measure on X s.t. the measure of every ball is finite. Then μ is σ -finite and for any Borel set $A \subset X$ and any $\epsilon > 0$ there exist an open set U and a closed set F s.t.

$$F \subset A \subset U \quad \text{and} \quad \mu(U \setminus F) < \epsilon. \quad (*)$$

In particular, μ is outer regular. If, furthermore, X is complete and separable, then μ is also inner regular.

Proof

σ -finiteness is clear.

Turning to (b), assume first that μ is finite.
Then (b) $\forall \varepsilon > 0$ is equivalent to

$$\sup_{F \subset A} \mu(F) = \mu(A) = \inf_{A \subset U} \mu(U). \quad (**)$$

The set of sets satisfying (**) forms a σ -algebra containing all closed sets (as they are G_δ -sets), hence it coincides with the σ -algebra of Borel sets. Assume now that μ is infinite. Choose $x_0 \in X$.

Put $A_n = \{x \in X \mid n-1 \leq d(x, x_0) < n\}$

By the finite measure case ~~there exist~~ (applied to $A_n \subset B_{n+1/2}(x_0)$) there exist sets F_n and U_n that are closed, resp. open, s.t. $\mu(U_n \setminus F_n) < \frac{\varepsilon}{2^n}$ and $F_n \subset A_n \subset U_n$. Then

F_n is closed in X , as $F_n \subset B_{n+1/2}(x_0)$.

Consider $F = \bigcup_{n=1}^{\infty} F_n$, $U = \bigcup_{n=1}^{\infty} U_n$. Then F is closed, U is open and $\mu(U \setminus F) < \varepsilon$.

In particular, $\mu(A) \leq \mu(U) + \varepsilon$, so μ is outer regular.

Assume now that X is complete and separable. For any Borel set A we have

$$\mu(A) = \sup_{F \subset A} \mu(F) = \sup_{F \subset A} \sup_{F \text{ closed}} \mu(F).$$

It follows that in proving inner regularity we may assume that A is closed and μ is finite.

~~Then~~ Then A is itself separable and complete. Therefore it suffices to consider $A = X$.

Fix $\varepsilon > 0$. For every n we can cover X by countably many ^{open} balls $U_{n,k}$, $k \geq 1, 2, \dots$, of radius 2^{-n} . Then, for every n , we

can choose W_n s.t. $\mu(X \setminus W_n) < \varepsilon 2^{-n}$. Consider the
 set $W = \bigcap_{n=1}^{\infty} W_n$. Then $\mu(X \setminus W) < \varepsilon$.
 For every n the set W is contained in W_n ,
 so it can be covered by finitely many balls of radius 2^{-n} .
 Therefore $\forall \delta > 0$ there exists a finite δ -net in W .
 Hence $K = \overline{W}$ is compact and $\mu(X \setminus K) < \varepsilon$. $\square$

Instead of conditions involving the metric, it is often convenient to have only topological properties.

Recall that a metric space (X, d) is called locally compact if $\forall x \in X$ $\exists$ a ball $B_R(x) = \{y: d(y, x) < R\}$ s.t. $B_R(x)$ is compact. (X, d) is called separable if it has a dense ~~open~~ countable subset.

Theorem

Assume (X, d) is a locally compact separable metric space, μ is a Borel measure on X s.t. $\mu(K) < \infty$ for any compact $K \subset X$. Then μ is regular and σ -finite.

Proof

Let us show first that there is a sequence of open sets U_n s.t. $\bigcup_{n=1}^{\infty} U_n = X$, $\overline{U_n}$ is compact and $\overline{U_n} \subset U_{n+1}$ for all n .

Let $(x_n)_{n=1}^{\infty}$ be a dense sequence in X .

Consider all open balls $B_R(x_n)$ s.t.

~~where~~ $R > 0$ is rational and $B_R(x_n)$ is compact.

Let $B_1, B_2, \dots$ be all these balls. Then

$\bigcup_{n=1}^{\infty} B_n = X$; if $x \in X$, we find $R > 0$ s.t.

$B_R(x)$ is compact, then find x_n s.t. $d(x, x_n) < \frac{R}{3}$

and then find $B_m = B_r(x_n)$ with $r > \frac{R}{3}$,
since $B_s(x_n) \subset B_R(x)$ is compact for all $s \leq \frac{2R}{3}$;

then $x \in B_r(x_n) = B_m$.

Now define U_n by induction as follows!

Put $U_1 = B_1$. If $U_1, \dots, U_n$ are constructed, choose $m_{n+1} \geq n+1$ s.t. $\bar{U}_n \subset B_1 \cup \dots \cup B_{m_{n+1}}$

and let $U_{n+1} = B_1 \cup \dots \cup B_{m_{n+1}}$.

~~The sequence U_n is increasing and $\bigcup U_n = X$.~~

Note that we have $\mu(U_n) \leq \mu(\bar{U}_n) < \infty$.

Now, take any Borel set $A \subset X$.

As $A \cap \bar{U}_n \uparrow A$, we have $\mu(A) = \sup_n \mu(A \cap \bar{U}_n)$.

By the previous theorem applied to the space $\bar{U}_n$,

$$\mu(A \cap \bar{U}_n) = \sup_{\substack{F \subset A \cap \bar{U}_n \\ \text{closed}}} \mu(F)$$

As a closed ~~subset of $\bar{U}_n$~~ ^{subset} ~~of $\bar{U}_n$~~ ^{compact} at $\bar{U}_n$ is compact, this shows that μ is inner regular.

~~Applying the previous theorem to U_n~~ Applying the previous theorem to U_n , we can find an open

subset $V_n \subset U_n$ s.t.

$$A \cap U_n \subset V_n \text{ and } \mu(V_n \setminus A) < \frac{\epsilon}{2^n}.$$

Consider $V = \bigcup_{n=1}^{\infty} V_n$. Then V is open, $A \subset V$

$$\text{and } \mu(V \setminus A) \leq \mu\left(\bigcup_{n=1}^{\infty} (V_n \setminus A)\right) \leq \sum_{n=1}^{\infty} \mu(V_n \setminus A) < \epsilon.$$

Hence μ is outer regular. $\square$

The measures μ as in this theorem are called Radon measures.

Warning For more general spaces the definition of a Radon measure is more complicated.