

Complex and signed measures

①

Assume $(X, \mathcal{B})$ is a measurable space.

Def

A complex measure on $(X, \mathcal{B})$ is a map $\nu: \mathcal{B} \rightarrow \mathbb{C}$ s.t. $\nu(\emptyset) = 0$ and $\nu(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \nu(A_n)$ for any disjoint $A_n \in \mathcal{B}$, where the series is assumed to be absolutely convergent. If ν takes values in $\mathbb{R}$, then ν is called a finite signed measure.

Remark

More generally, a signed measure is allowed to take values in $\mathbb{R} \cup \{+\infty\}$ or $\mathbb{R} \cup \{-\infty\}$.

Given a complex measure ν on $(X, \mathcal{B})$, its total variation is the map

$$|\nu|: \mathcal{B} \rightarrow [0, +\infty]$$

defined by

$$|\nu|(A) = \sup \left\{ \sum_{n=1}^{\infty} |\nu(A_n)| : A = \bigcup_{n=1}^{\infty} A_n, \right. \\ \left. A_n \in \mathcal{B}, A_n \cap A_m = \emptyset \text{ for } n \neq m \right\}$$

Proposition

$|\nu|$ is a finite measure on $(X, \mathcal{B})$.

Proof

Let us first show that $|\nu|$ is a measure.

As $\nu(\emptyset) = 0$, we have $|\nu|(\emptyset) = 0$.

For σ -additivity, take $A = \bigcup_{n=1}^{\infty} A_n$,
 $A_n \in \mathcal{B}$, $A_n \cap A_m = \emptyset$ for $n \neq m$.

If $A = \bigcup_{k=1}^N B_k$, $B_k \in \mathcal{B}$, $B_k \cap B_l = \emptyset$ for $k \neq l$,

then

$$\begin{aligned} \sum_{k=1}^N |V(B_k)| &= \sum_{k=1}^N \left| \sum_{n=1}^{\infty} V(B_k \cap A_n) \right| \\ &\leq \sum_{n=1}^{\infty} \sum_{k=1}^N |V(B_k \cap A_n)| \\ &\leq \sum_{n=1}^{\infty} |V(A_n)|. \end{aligned}$$

Taking the supremum over all such decompositions $A = \bigcup_{k=1}^N B_k$, we get

$$|V|(A) \leq \sum_{n=1}^{\infty} |V|(A_n).$$

To prove the opposite inequality we may assume $|V|(A) < \infty$. Then

$|V|(A_n) \leq |V|(A) < \infty$ for all n . It suffices to show that $\sum_{n=1}^N |V|(A_n) \leq |V|(A)$ for all N .

Fix $\varepsilon > 0$ and choose decompositions

$A_n = \bigcup_{k=1}^{m_n} B_{nk}$, $B_{nk} \in \mathcal{B}$, $B_{nk} \cap B_{nl} = \emptyset$ for $k \neq l$,
 such that $\sum_{k=1}^{m_n} |V(B_{nk})| > |V|(A_n) - \varepsilon$.

Then

$$\begin{aligned} |V|(A) &\geq \sum_{n=1}^N \sum_{k=1}^{m_n} |V(B_{nk})| + |V(A \setminus \bigcup_{n=1}^N A_n)| \\ &> \sum_{n=1}^N (|V|(A_n) - \varepsilon) = \sum_{n=1}^N |V|(A_n) - N\varepsilon. \end{aligned}$$

As $\varepsilon > 0$ was arbitrary, we conclude that

$$|V|(A) \geq \sum_{n=1}^{\infty} |V|(A_n).$$

It remains to check that $|V|(X) < \infty$. ②

As $|V|(X) \leq |Re V|(X) + |Im V|(X)$, we can consider separately $Re V$ and $Im V$.
In other words, it is enough to consider finite signed measures.

Assume $|V|(X) = +\infty$. We claim that there exist $A, B \in \mathcal{B}$ s.t.

$$X = A \cup B, A \cap B = \emptyset, |V|(A) = +\infty, |V|(B)| \geq 1.$$

To see this, find a decomposition

$$X = \bigcup_{n=1}^{\infty} A_n, A_n \in \mathcal{B}, A_n \cap A_m = \emptyset, \text{ s.t.}$$

$$\sum_{n=1}^{\infty} |V(A_n)| \geq |V(X)| + 2 = \left| \sum_{n=1}^{\infty} V(A_n) \right| + 2.$$

Consider $I = \{i : 1 \leq i \leq N, V(A_i) \geq 0\}$ and

$J = \{j : 1 \leq j \leq N, V(A_j) < 0\}$: we then have

$$\sum_{k \in I} V(A_k) - \sum_{k \in J} V(A_k) \geq \left| \sum_{k \in I} V(A_k) - \sum_{k \in J} V(A_k) \right| + 2$$

This implies that

$$\sum_{k \in I} V(A_k) \geq 1 \quad \text{and} \quad \sum_{k \in J} V(A_k) \leq -1.$$

Let $B_0 = \bigcup_{k \in I} A_k$, $B_1 = \bigcup_{k \in J} A_k$. Then

$$X = B_0 \cup B_1, B_0 \cap B_1 = \emptyset, |V(B_0)| \geq 1, |V(B_1)| \geq 1.$$

As $|V|(X) = |V|(B_0) + |V|(B_1)$, we must

have $|V|(B_0) = +\infty$ or $|V|(B_1) = +\infty$.

If $|V|(B_0) = +\infty$, we let $A = B_0$, $B = B_1$.

If $|V|(B_0) < \infty$, we let $A = B_1$, $B = B_0$.

This proves the claim.

we can apply the claim to the restriction of ν to A , and so on. Therefore by induction we can construct disjoint measurable set $B_1, B_2, \dots$ s.t.

$$|\nu(B_n)| \geq 1 \quad \forall n \quad \text{and} \quad |\nu|(X \setminus \bigcup_{n=1}^N B_n) = +\infty \quad \forall N$$

But then $\nu(\bigcup_{n=1}^{\infty} B_n) = \sum_{n=1}^{\infty} \nu(B_n)$, which does not make sense, as the series is not absolutely convergent. Hence $|\nu|(X) < \infty$. □

Example

Consider a measure space $(X, \mathcal{B}, \mu)$ and take $f \in L^1(X, d\mu)$. Define

$$\nu(A) = \int_A f d\mu.$$

Then ν is a complex measure on $(X, \mathcal{B})$, since this is true for $f \geq 0$ and general f can be written as a linear combination of positive ones. We write $d\nu = f d\mu$.

We then have $d|\nu| = |f| d\mu$, that is,

$$|\nu|(A) = \int_A |f| d\mu.$$

To see this, consider the measure ω defined by $d\omega = |f| d\mu$. We want to show that $|\nu| = \omega$.

Let $A = \bigcup_{n=1}^{\infty} A_n$, $A_n \cap A_m = \emptyset$, then

$$\begin{aligned} \sum_{n=1}^{\infty} |\nu(A_n)| &= \sum_{n=1}^{\infty} \left| \int_{A_n} f d\mu \right| \leq \sum_{n=1}^{\infty} \int_{A_n} |f| d\mu \\ &= \sum_{n=1}^{\infty} \omega(A_n) = \omega(A). \end{aligned}$$

Therefore $|V| \leq \omega$.

②

To prove the equality, assume first that f is simple, so $f = \sum_{n=1}^{\infty} c_n \chi_{A_n}$,

$c_n \in \mathbb{C}$, $A_n \in \mathcal{B}$, $A_n \cap A_m = \emptyset$ for $n \neq m$.

Then, for every $A \in \mathcal{B}$,

$$\begin{aligned} |V|(A) &\geq \sum_{n=1}^{\infty} |V(A \cap A_n)| = \sum_{n=1}^{\infty} |c_n| \mu(A \cap A_n) \\ &= \sum_{n=1}^{\infty} \int_{A \cap A_n} |f| d\mu = \int_A |f| d\mu = \omega(A). \end{aligned}$$

Thus, $|V| \geq \omega$, hence $|V| = \omega$.

For general f , choose simple functions $f_n \in L^1(X, d\mu)$ s.t. $\|f - f_n\|_1 \xrightarrow{n \rightarrow \infty} 0$. Consider the corresponding complex measures ν_n , $d\nu_n = f_n d\mu$. Fix $A \in \mathcal{B}$. For every $B \in \mathcal{B}$, we have

$$|V(B)| \leq |V - \nu_n|(B) + |\nu_n(B)|.$$

This implies that

$$|V|(A) \leq |V - \nu_n|(A) + |\nu_n(A)|.$$

$$\text{Similarly, } |\nu_n|(A) \leq |V - \nu_n|(A) + |V|(A).$$

$$\begin{aligned} \text{Therefore } | |V|(A) - |\nu_n|(A) | &\leq |V - \nu_n|(A) \leq \int_A |f - f_n| d\mu \\ &\leq \|f - f_n\|_1 \xrightarrow{n \rightarrow \infty} 0. \end{aligned}$$

It follows that

$$|V|(A) = \lim_{n \rightarrow \infty} |\nu_n|(A) = \lim_{n \rightarrow \infty} \int_A |f_n| d\mu = \int_A |f| d\mu$$

(since $||f| - |f_n|| \leq |f - f_n|$.) This completes the proof.

Def

If $(X, \mathcal{B}, \mu)$ is a measure space, ν is a complex measure on $(X, \mathcal{B})$, then we say that ν is absolutely continuous w.r.t. μ and write $\nu \ll \mu$, if $\nu(A) = 0$ whenever $A \in \mathcal{B}, \mu(A) = 0$. Equivalently, $|\nu| \ll \mu$.

Theorem (Radon-Nikodym theorem for complex measures)

Assume $(X, \mathcal{B}, \mu)$ is a measure space, ν is a complex measure on $(X, \mathcal{B})$, $\nu \ll \mu$. Then there is a unique $f \in L^1(X, d\mu)$ s.t.
 $d\nu = f d\mu$.

Proof

Existence By considering separately $\operatorname{Re} \nu$ and $\operatorname{Im} \nu$, we may assume that ν is a finite signed measure. Then
 $\nu = \nu_+ - \nu_-$, where $\nu_+ = \frac{|\nu| + \nu}{2}$, $\nu_- = \frac{|\nu| - \nu}{2}$,
are positive measures, since $|\nu(A)| \leq |\nu|(A)$.
Clearly, $\nu_+, \nu_- \ll \mu$. Therefore the proof reduces to the case when ν is positive, in which case we already know the result:
we take $f = \frac{d\nu}{d\mu}$; note that $\int_X f d\mu = \nu(X) < \infty$,
so $f \in L^1(X, d\mu)$.

Uniqueness It suffices to show that if $\nu = 0$ and $d\nu = f d\mu$, then $f = 0$ (μ -a.e.).
This is true, for example, because
 $\int_X |f| d\mu = |\nu|(X) = 0$.