

Decomposition theorems

①

Def

Two measures ν and μ on a measurable space $(X, \mathcal{B})$ are called mutually singular, or we say that ν is singular w.r.t. μ , if there is $N \in \mathcal{B}$ s.t. $\nu(N^c) = 0, \mu(N) = 0$. We then write $\nu \perp \mu$.

Theorem (Lebesgue decomposition theorem)

Assume ν, μ are σ -finite measures on $(X, \mathcal{B})$. Then there exist unique measures ν_a and ν_s s.t.

$$\nu = \nu_a + \nu_s, \quad \nu_a \ll \mu, \quad \nu_s \perp \mu.$$

Proof

As we showed earlier, there exist $N \in \mathcal{B}$ and measurable $f: X \rightarrow [0, +\infty)$ s.t. $\mu(N) = 0$ and

$$\nu(A) = \nu(A \cap N) + \int_A f d\mu \quad \forall A \in \mathcal{B}.$$

We then define

$$\nu_a(A) = \int_A f d\mu, \quad \nu_s(A) = \nu(A \cap N).$$

Assume we have another such decomposition $\nu = \tilde{\nu}_a + \tilde{\nu}_s$. Let $\tilde{N} \in \mathcal{B}$

be s.t. $\mu(\tilde{N}) = 0$ and $\tilde{\nu}_s(\tilde{N}^c) = 0$.

Then $\mu(N \cup \tilde{N}) = 0$ and $\nu_s((N \cup \tilde{N})^c) = 0$, $\tilde{\nu}_s((N \cup \tilde{N})^c) = 0$. Hence, for every $A \in \mathcal{B}$,

$$\begin{aligned}
 \nu_S(A) &= \nu_S(A \cap (N \cup \tilde{N})) \\
 &= \nu_S(A \cap (N \cup \tilde{N})) + \nu_a(A \cap (N \cup \tilde{N})) \\
 &= \nu(A \cap (N \cup \tilde{N})),
 \end{aligned}$$

and for the same reasons

$$\tilde{\nu}_S(A) = \nu(A \cap (N \cup \tilde{N})).$$

Thus, $\nu_S = \tilde{\nu}_S$.

In a similar way,

$$\begin{aligned}
 \nu_a(A) &= \nu_a(A \cap (N \cup \tilde{N})^c) \\
 &= \nu_a(A \cap (N \cup \tilde{N})^c) + \nu_S(A \cap (N \cup \tilde{N})^c) \\
 &= \nu(A \cap (N \cup \tilde{N})^c),
 \end{aligned}$$

$$\tilde{\nu}_a(A) = \nu(A \cap (N \cup \tilde{N})^c).$$

Hence $\nu_a = \tilde{\nu}_a$. □

Theorem (polar decomposition of complex measures)

Assume ν is a complex measure on $(X, \mathcal{B})$. Then there exist a finite measure μ on $(X, \mathcal{B})$ and a measurable function $f: X \rightarrow \mathbb{C}$ s.t. $d\nu = f d\mu$. If $(\tilde{\mu}, \tilde{f})$ is another such pair, then $\tilde{\mu} = \mu$ and $\tilde{f} = f$ μ -a.e.

Proof

Take $\mu = |\nu|$. Then $\nu \ll |\nu|$.

Hence by the Radon-Nikodym theorem

for complex measures, we have $d\nu = f d\mu$ for a unique $f \in L^1(X, d\mu)$. ②

Then $d|\nu| = |f| d\mu = |f| d|\nu|$.

Hence $|f| = 1$ (μ -a.e.). By viewing f as a function, we can therefore assume that $f: X \rightarrow \mathbb{T}$.

Assume we have another such decomposition $d\nu = \tilde{f} d\tilde{\mu}$. Then $d|\nu| = |\tilde{f}| d\tilde{\mu} = d\tilde{\mu}$, that is, $\tilde{\mu} = \mu = |\nu|$. Then $d\nu = \tilde{f} d\mu$, hence $f = \tilde{f}$ μ -a.e. □

For signed measures this leads to the following.

Theorem (Hahn decomposition theorem)

Assume ν is a finite signed measure on $(X, \mathcal{B})$. Then there exist $P, N \in \mathcal{B}$ s.t.

$$X = P \cup N, \quad P \cap N = \emptyset,$$

$$\nu(A \cap P) \geq 0, \quad \nu(A \cap N) \leq 0 \quad \forall A \in \mathcal{B}.$$

Moreover, then $|\nu|(A) = \nu(A \cap P) - \nu(A \cap N)$, and if $X = \tilde{P} \cup \tilde{N}$ is another such decomposition, then

$$|\nu|(P \Delta \tilde{P}) = |\nu|(N \Delta \tilde{N}) = 0.$$

Proof

Consider the polar decomposition

$$d\nu = f |d\nu|, \quad f: X \rightarrow \mathbb{T}.$$

Since ν is real-valued, we have $\operatorname{Im} f = 0$
 $|d\nu|$ -a.e. We may therefore assume that

f takes values in $\mathbb{T} \cap \mathbb{R} = \{-1, 1\}$.

Let $P = \{x: f(x) = 1\}$, $N = \{x: f(x) = -1\}$.

Then

$$\nu(A \cap P) = \int_{A \cap P} f |d\nu| = \int_{A \cap P} 1 |d\nu| = \nu(A \cap P) \geq 0,$$

$$\nu(A \cap N) = \int_{A \cap N} f |d\nu| = \int_{A \cap N} (-1) |d\nu| = -\nu(A \cap N) \leq 0.$$

This gives the required decomposition.

Moreover, we see that

$$\begin{aligned} \nu(A \cap P) - \nu(A \cap N) &= \nu(A \cap P) + \nu(A \cap N) \\ &= \nu(A). \end{aligned}$$

Assume $X = \tilde{P} \cup \tilde{N}$ another decomposition
as in the statement of the theorem.

For $A \subset P \setminus \tilde{P}$ ($A \in \mathcal{B}$) we have

$$\nu(A) = \nu(A \cap P) \geq 0,$$

$$\nu(A) = \nu(A \cap \tilde{N}) \leq 0.$$

It follows that $\nu(A) = 0$. Hence

$$\nu(P \setminus \tilde{P}) = 0. \quad \text{Similarly } \nu(\tilde{P} \setminus P) = 0.$$

$$\text{Hence } \nu(P \Delta \tilde{P}) = \nu(P \setminus \tilde{P}) + \nu(\tilde{P} \setminus P) = 0,$$

$$\nu(N \Delta \tilde{N}) = \nu(N^c \Delta \tilde{N}^c) = \nu(P \Delta \tilde{P}) = 0. \quad \square$$

Corollary (Jordan decomposition theorem) ③

Assume ν is a finite signed measure on $(X, \mathcal{B})$. Then there exist unique finite measures ν_+, ν_- on $(X, \mathcal{B})$ s.t.

$$\nu = \nu_+ - \nu_- \quad \text{and} \quad \nu_+ \perp \nu_-.$$

Moreover, then $|\nu| = \nu_+ + \nu_-$, hence

$$\nu_+ = \frac{|\nu| + \nu}{2}, \quad \nu_- = \frac{|\nu| - \nu}{2}.$$

Proof

Consider the Hahn decomposition

$$X = P \sqcup N. \quad \text{Define}$$

$$\nu_+(A) = \nu(A \cap P), \quad \nu_-(A) = -\nu(A \cap N).$$

Then by the properties of the Hahn decomposition ~~xxxxxxxx~~ we have

$$\nu = \nu_+ - \nu_-, \quad \nu_+, \nu_- \geq 0, \quad |\nu| = \nu_+ + \nu_-,$$

and $\nu_+ \perp \nu_-$ by definition.

Assume we have another such decomposition $\nu = \tilde{\nu}_+ - \tilde{\nu}_-, \quad \tilde{\nu}_+ \perp \tilde{\nu}_-.$

Let $\tilde{N} \in \mathcal{B}$ be such that $\tilde{\nu}_+(\tilde{N}) = 0$, $\tilde{\nu}_-(\tilde{N}^c) = 0$. Put $\tilde{P} = X \setminus \tilde{N}$. Then

$X = \tilde{P} \cup \tilde{N}$ is again a Hahn decomposition.

$$\text{In particular, } |\nu|(A) = \nu(A \cap \tilde{P}) - \nu(A \cap \tilde{N})$$

$$= \tilde{\nu}_+(A \cap \tilde{P}) + \tilde{\nu}_-(A \cap \tilde{N}) = \tilde{\nu}_+(A) - \tilde{\nu}_-(A).$$

Thus $|\nu| = \tilde{\nu}_+ - \tilde{\nu}_-$. It follows that

$$\tilde{v}_+ = \frac{1 \pm \gamma}{2} = v_+, \quad \tilde{v}_- = \frac{1 \pm \gamma}{2} = v_-.$$