

Duals of L^p -spaces

(1)

Assume $(X, \mathcal{B}, \mu)$ is a measure space, $1 \leq p < \infty$.
What is the dual of $L^p(X, \mu)$?

When does a measurable function $g: X \rightarrow \mathbb{C}$
define a bounded linear functional on $L^p(X, \mu)$ by
$$\varphi(f) = \int_X fg d\mu$$
?

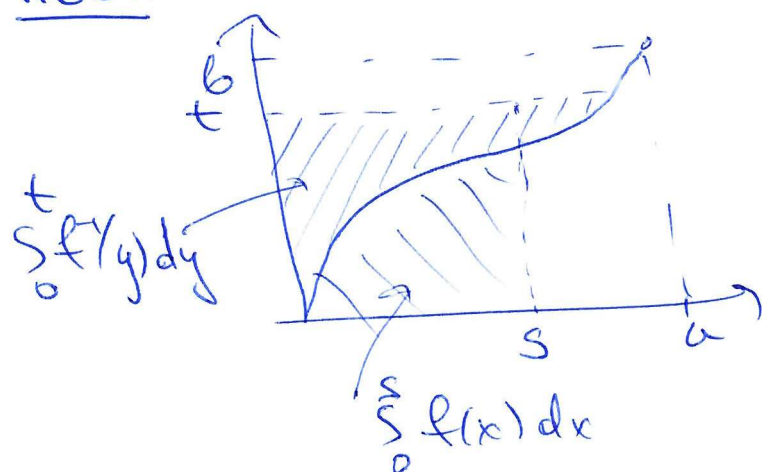
Young's inequality

Assume $f: [0, a] \rightarrow [0, b]$ is a strictly increasing
continuous function, $f(0)=0$, $f(a)=b$. Then,
for all $s \in [0, a]$ and $t \in [0, b]$, we have

$$st \leq \int_0^s f(x) dx + \int_0^t f^{-1}(y) dy,$$

and the equality holds if and only if $t = f(s)$.

Proof



Compare areas

Apply this to $f(s) = s^{p-1}$. Then $f^{-1}(t) = t^{q-1}$,
where q is the Hölder conjugate of p :

$(p-1)(q-1)=1$, that is, $\frac{1}{p} + \frac{1}{q} = 1$. We get

$$st \leq \int_0^s x^{p-1} dx + \int_0^t y^{q-1} dy = \frac{s^p}{p} + \frac{t^q}{q}.$$

Hölder inequality

If $f \in L^p(X, d\mu)$, $g \in L^q(X, d\mu)$, $1 < p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$,
then $fg \in L^1(X, d\mu)$ and
$$\|fg\|_1 \leq \|f\|_p \|g\|_q$$

Proof

Assume $\|f\|_p > 0$, $\|g\|_q > 0$, otherwise $fg = 0$ (a.e.)
we have

$$\frac{|fg|}{\|f\|_p \|g\|_q} \leq \frac{1}{p} \frac{|f|^p}{\|f\|_p^p} + \frac{1}{q} \cdot \frac{|g|^q}{\|g\|_q^q}$$

Integrating over X we get

$$\begin{aligned} \frac{\|fg\|_1}{\|f\|_p \|g\|_q} &\leq \frac{1}{p} \int_X \frac{|f|^p}{\|f\|_p^p} d\mu + \frac{1}{q} \int_X \frac{|g|^q}{\|g\|_q^q} d\mu \\ &= \frac{1}{p} \frac{\|f\|_p^p}{\|f\|_p^p} + \frac{1}{q} \frac{\|g\|_q^q}{\|g\|_q^q} \\ &= \frac{1}{p} + \frac{1}{q} = 1, \end{aligned}$$

so $\|fg\|_1 \leq \|f\|_p \|g\|_q$. □

It follows that every $g \in L^q(X, d\mu)$
defines a bounded linear functional

$$\ell_g : L^p(X, d\mu) \rightarrow \mathbb{C}, \quad \ell_g(f) = \int_X fg d\mu,$$

and $\|\ell_g\| \leq \|g\|_q$.

The same makes sense for $p=1$, $q=\infty$
and $p=\infty$, $q=1$, when μ is σ -finite,

as
$$\int_X |fg| d\mu \leq \int_X |f| d\mu \|g\|_\infty = \|f\|_1 \|g\|_\infty$$

(2)

Lemma

Assume $1 \leq p \leq \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, and μ is σ -finite if $p=1$ or $p=\infty$. For $g \in L^q(X, d\mu)$ consider $\ell_g \in L^p(X, d\mu)^*$. Then

$$\|\ell_g\| = \|g\|_q.$$
Proof

We may assume $\|g\|_q > 0$, otherwise $\ell_g = 0$. Consider three cases:

1) $1 < p < \infty$, so $1 < q < \infty$.

We already know that $\|\ell_g\| \leq \|g\|_q$.

To prove the opposite inequality, consider

$$f = \frac{\overline{g}}{|g|} |g|^{q-1}. \text{ Then}$$

$$|f|^p = |g|^{(q-1)p} = |g|^{q(1-\frac{1}{q})p} = |g|^q,$$

so $\|f\|_p^p = \|g\|_q^q$. We have

$$\ell_g(f) = \int_X |g|^q d\mu = \|g\|_q^q. \text{ Hence}$$

$$\begin{aligned} \|\ell_g\| &\geq \frac{|\ell_g(f)|}{\|f\|_p} = \frac{\|g\|_q^q}{\|g\|_q^{q/p}} = \|g\|_q^{q - \frac{q}{p}} \\ &= \|g\|_q^{q(1-\frac{1}{p})} = \|g\|_q. \end{aligned}$$

2) $p=1$, so $q=\infty$.

Take $0 < c < \|g\|_\infty$.

Then $\mu(\{x: |g(x)| > c\}) > 0$.

As μ is σ -finite, so $X = \bigcup_{n=1}^\infty X_n$, $\mu(X_n) < \infty$,

we can find $A = X_n \cap \{x: |g(x)| > c\}$
 s.t. $\mu(A) > 0$ and $\mu(A) < \infty$.

Consider $f = \frac{g}{|g|} \chi_A$. Then $\|f\|_1 = \mu(A)$,

$$\ell_g(f) = \int_X |g| \chi_A d\mu = \int_A |g| d\mu > c \mu(A).$$

Hence $\|\ell_g\| \geq \frac{|\ell_g(f)|}{\|f\|_1} > c$. It follows

that $\|\ell_g\| \geq \|g\|_\infty$.

3) $p = \infty$, so $q = 1$.

Define $f(x) = \begin{cases} \frac{g(x)}{|g(x)|} & \text{if } g(x) \neq 0, \\ 0 & \text{otherwise.} \end{cases}$

Then $\|f\|_\infty = 1$ and

$$\ell_g(f) = \int_X |g| d\mu = \|g\|_1.$$

Hence $\|\ell_g\| \geq \|g\|_1$. □

Therefore we can view $L^q(X, d\mu)$ as
 a subspace of $L^p(X, d\mu)^*$ using the
 isometric embedding

$$L^q(X, d\mu) \hookrightarrow L^p(X, d\mu)^*, \quad g \mapsto \ell_g.$$

Theorem

Assume $(X, \mathcal{B}, \mu)$ is a σ -finite measure
 space, $1 \leq p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$. Then

$$L^p(X, d\mu)^* = L^q(X, d\mu).$$

Remark / warning

③

This is usually not true for $p = \infty$.

Proof of the theorem

Assume first that $\mu(X) < \infty$.

Take $\psi \in L^p(X, d\mu)$. We want to show that $\psi = f g$ for some $g \in L^q(X, d\mu)$.

As $\mu(X) < \infty$, for every $A \in \mathcal{B}$ we have $\pi_A \in L^p(X, d\mu)$ and $\|\pi_A\|_p = \mu(A)^{1/p}$.

Define $\nu : \mathcal{B} \rightarrow \mathbb{C}$ by $\nu(A) = \psi(\pi_A)$.

This is a complex measure on $(X, \mathcal{B})$.

Indeed, clearly $\nu(\emptyset) = 0$. Assume

$A = \bigcup_{n=1}^{\infty} A_n$, $A_n \in \mathcal{B}$, $A_n \cap A_m = \emptyset$ for $n \neq m$.

Then

$$\begin{aligned} |\nu(A) - \sum_{n=1}^N \nu(A_n)| &= |\psi(\pi_A) - \sum_{n=1}^N \psi(\pi_{A_n})| \\ &= |\psi(\pi_{A \setminus \bigcup_{n=1}^N A_n})| \\ &\leq \|\psi\| \|\pi_{A \setminus \bigcup_{n=1}^N A_n}\|_p \\ &= \|\psi\| \mu(A \setminus \bigcup_{n=1}^N A_n)^{1/p} \xrightarrow{N \rightarrow \infty} 0. \end{aligned}$$

Therefore $\nu(A) = \sum_{n=1}^{\infty} \nu(A_n)$.

We have $\nu \ll \mu$, since $\pi_A = 0$ in $L^p(X, d\mu)$ if $\mu(A) = 0$. By the Radon-Nikodym theorem for complex measures we get

$g \in L^1(X, d\mu)$ s.t. $\nu(A) = \int_A g d\mu$. Then

$$\psi(f) = \int_X fg d\mu \quad \text{for all simple functions } f.$$

Consider $X_n = \{x: |g(x)| \leq n\}$.

Then $g|_{X_n} \in L^q(X_n, d\mu)$, and

$\psi = \ell_{g|_{X_n}}$ on simple functions in $L^p(X_n, d\mu)$

As such functions are dense in $L^p(X_n, d\mu)$, we set $\psi = \ell_{g|_{X_n}}$ on $L^p(X_n, d\mu)$. Hence

$$\|g|_{X_n}\|_q = \|\psi|_{L^p(X_n, d\mu)}\| \leq \|\psi\|.$$

We have $\|g|_{X_n}\|_q \xrightarrow{n \rightarrow \infty} \|g\|_q$. Indeed, if $1 < q < \infty$, this is true by the monotone convergence theorem, and if $q = \infty$ this is true by definition.

Thus $\|g\|_q \leq \|\psi\|$, so $g \in L^q(X, d\mu)$.

As $\psi = \ell_g$ on simple functions and they are dense in $L^p(X, d\mu)$, we set $\psi = \ell_g$. This finishes the proof for finite measures.

Assume now that μ is σ -finite.

Thus, $X_n \uparrow X$ for some $X_n \in \mathcal{B}$, $\mu(X_n) < \infty$. Take $\psi \in L^p(X, d\mu)^*$.

Applying the first part of the proof to $\varphi|_{L^p(X_n, d\mu)}$, we get functions $g_n \in L^p(X_n, d\mu)$ s.t. $\varphi = l_{g_n}$ on $L^p(X_n, d\mu)$. Then, if $n > m$, we have $l_{g_n}|_{X_m} = l_{g_m}$, hence $g_n|_{X_m} = g_m$ (μ -a.e.).

It follows that we can define a measurable function g s.t. $g|_{X_n} = g_n$ (μ -a.e.) for all n . Then

$$\|g\|_q = \lim_{n \rightarrow \infty} \|g|_{X_n}\|_q = \lim_{n \rightarrow \infty} \|\varphi|_{L^p(X_n, d\mu)}\| \leq \|\varphi\|,$$

so $g \in L^q(X, d\mu)$.

We also have $\varphi = l_g$ on $\bigcup_{n=1}^{\infty} L^p(X_n, d\mu)$.

But if $f \in L^p(X, d\mu)$, then ~~$f|_{X_n} \in L^p(X_n, d\mu)$~~

$f|_{X_n} \in L^p(X_n, d\mu)$ and $f|_{X_n} \rightarrow f$ in $L^p(X, d\mu)$.

Therefore $\bigcup_{n=1}^{\infty} L^p(X_n, d\mu)$ is dense in $L^p(X, d\mu)$.

Hence $\varphi = l_g$. □