

Riesz-Markov theorem

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Assume (X, d) is a locally compact metric space.

A linear functional $\varphi: C_c(X) \rightarrow \mathbb{C}$ is called positive if $\varphi(f) \geq 0$ for all $f \geq 0$.

Theorem (Riesz-Markov)

If $\varphi: C_c(X) \rightarrow \mathbb{C}$ is a positive linear functional, where (X, d) is a locally compact metric space, then there exists a Borel measure μ on X s.t. $\mu(K) < \infty$ for every compact $K \subset X$ and

$$\varphi(f) = \int_X f d\mu \quad \forall f \in C_c(X).$$

If X is separable, then such a measure μ is unique.

For the proof we need two auxiliary results.

Urysohn's lemma

Assume (X, d) is a metric space, $A, B \subset X$ are disjoint closed subsets. Then there exists a continuous function $f: X \rightarrow [0, 1]$ s.t. $f \equiv 1$ on A and $f \equiv 0$ on B .

Proof

$$d(x, B)$$

Define $f(x) = \frac{d(x, B)}{d(x, A) + d(x, B)}$

Note that this is well-defined, since

$d(x, A) + d(x, B) > 0$ for all x , as
 $\{x: d(x, A) = d(x, B) = 0\} = A \cap B = \emptyset$.

Lemma

Assume (X, d) is a compact metric space, $\mathcal{U} = (U_i)_{i=1}^n$ is a finite open cover of X (so U_i are open and $\bigcup_{i=1}^n U_i = X$).

Then there exist functions $f_1, \dots, f_n$ in $C(X)$ s.t.

$$0 \leq f_i \leq 1, \text{ supp } f_i \subset U_i, \sum_{i=1}^n f_i(x) = 1 \quad \forall x.$$

Every such collection of functions is called a partition of unity subordinate to $\mathcal{U}$.

Proof

Let us show first that there exists an open set V_1 s.t. $\overline{V_1} \subset U_1$ and $X = V_1 \cup U_2 \cup \dots \cup U_n$.

For this, consider the closed set $K = X \setminus (U_2 \cup \dots \cup U_n) \subset U_1$. For every $x \in K$ we can find a ball $B_{r_x}(x)$ ($r_x > 0$) s.t. $\overline{B_{r_x}(x)} \subset U_1$. As K is compact, and $(B_{r_x}(x))_{x \in K}$ is an open cover of K , we can find a finite subcover $(B_{r_{x_k}}(x_k))_{k=1}^m$. Then put $V_1 = \bigcup_{k=1}^m B_{r_{x_k}}(x_k)$.

Using this construction $2n$ times (2)
 we can find open sets $V_1', \dots, V_n'$ and
 $V_1'', \dots, V_n''$ s.t.

$$\overline{V_i'} \subset V_i'', \quad \overline{V_i''} \subset U_i, \quad \bigcup_{i=1}^n V_i' = X.$$

By Urysohn's lemma, for every i ,
 we can find $f_i \in C(X)$ s.t.

$$0 \leq f_i \leq 1, \quad f_i \equiv 1 \text{ on } \overline{V_i'}, \quad f_i \equiv 0 \text{ on } X \setminus V_i''.$$

Then $\text{supp } f_i \subset \overline{V_i''} \subset U_i$.

$$\text{Define } g_i(x) = \frac{f_i(x)}{\sum_{j=1}^n f_j(x)}.$$

Proof of the Riesz-Markov theorem

Assume $\mu: C_c(X) \rightarrow \mathbb{R}$ is positive.

For open $U \subset X$, define

$$\mu(U) = \sup \{ \mu(f) : 0 \leq f \leq 1, \text{supp } f \subset U \}.$$

For a set $A \subset X$, define then

$$\mu^*(A) = \inf_{\substack{A \subset \bigcup U \\ U \text{ open}}} \mu(U).$$

Step 1 μ^* is an outer measure on X
 and $\mu^*(U) = \mu(U)$ for open U .

Observe first that if $V \subset U$ are open,
 then $\mu(V) \leq \mu(U)$. This implies that

$\mu^*(V) = V$ for all open V .

Obviously, $\mu^*(\emptyset) = \mu(\emptyset) = 0$.

We need to show that

$$\mu^*\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} \mu^*(A_n).$$

We may assume that $\mu^*(A_n) < \infty$ for all n , as otherwise there is nothing to prove.

Assume first that $A_n \subseteq U_n$ are open.

Take $0 \leq f \leq 1$ s.t. $\text{supp } f \subseteq \bigcup_{n=1}^{\infty} U_n$.
($f \in C_c(X)$)

As $\text{supp } f$ is compact, there is N s.t. $\text{supp } f \subseteq \bigcup_{n=1}^N U_n$. Let $\rho_1, \dots, \rho_N$ be a partition of unity in $C(\text{supp } f)$ subordinate to $(U_n \cap \text{supp } f)_{n=1}^N$. Define $f_n(x) = \begin{cases} \rho_n(x) f(x), & \text{if } x \in \text{supp } f, \\ 0 & \text{otherwise.} \end{cases}$

Note that the functions f_n are continuous. (Indeed, if $x \notin \text{supp } f$ or $x \in \text{int}(\text{supp } f)$, then $f(y) \rightarrow f(x)$ as $y \rightarrow x$, as 0 is continuous on $X \setminus \text{supp } f$ and ρ_n, f are continuous on $\text{int}(\text{supp } f)$. Assume now that $x \in (\text{supp } f) \setminus \text{int}(\text{supp } f)$, so $x \in \text{supp } f$, but $B_r(x) \setminus \text{supp } f \neq \emptyset \ \forall r > 0$. Then $f(x) = 0$ and $f(y) \rightarrow 0$ as $y \rightarrow x$, since $\lim_{y \rightarrow x, y \in \text{supp } f} \rho_n(y) f(y) = \rho_n(x) f(x) = 0$.)

Note also that

$$\text{supp } f_n \subset \text{supp } \mu_n \subset U_n.$$

Hence $\psi(f_n) \leq \mu(U_n)$. Therefore

$$\psi(f) = \sum_{n=1}^{\infty} \psi(f_n) \leq \sum_{n=1}^{\infty} \mu(U_n) \leq \sum_{n=1}^{\infty} \mu(U_n).$$

Taking the supremum over all f s.t.

$0 \leq f \leq 1$, $\text{supp } f \subset \bigcup_{n=1}^{\infty} U_n$, we get

$$\mu\left(\bigcup_{n=1}^{\infty} U_n\right) \leq \sum_{n=1}^{\infty} \mu(U_n).$$

For general A_n , fix $\varepsilon > 0$ and choose

open U_n s.t.

$$A_n \subset U_n \text{ and } \mu(U_n) < \mu^*(A_n) + \frac{\varepsilon}{2^n}.$$

Then

$$\begin{aligned} \mu^*\left(\bigcup_{n=1}^{\infty} A_n\right) &\leq \mu^*\left(\bigcup_{n=1}^{\infty} U_n\right) \leq \sum_{n=1}^{\infty} \mu(U_n) \\ &< \sum_{n=1}^{\infty} \left(\mu^*(A_n) + \frac{\varepsilon}{2^n}\right) \\ &= \sum_{n=1}^{\infty} \mu^*(A_n) + \varepsilon. \end{aligned}$$

As $\varepsilon > 0$ was arbitrary, we conclude that

$$\mu^*\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} \mu^*(A_n).$$

This completes Step 1.

Step 2 All Borel sets are Carathéodory measurable (with respect to μ^*).

It suffices to show that every open

set U is Carathéodory measurable. We need to check that

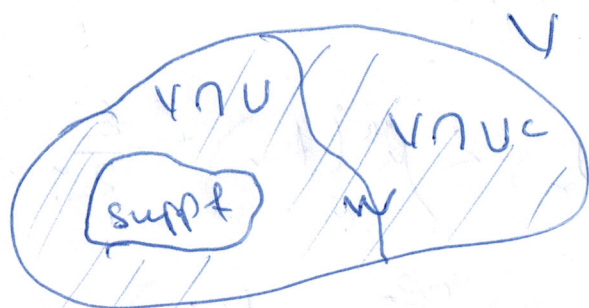
$$\mu^*(A) \geq \mu^*(A \cap U) + \mu^*(A \cap U^c) \quad \forall A \subset X.$$

Consider first an open set $A = V$.
We may assume that $\mu(V) < \infty$, as otherwise there is nothing to prove.

Fix $\varepsilon > 0$ and choose δ, ε_1 .

$$0 \leq \delta \leq 1, \text{ supp } \delta \subset V \cap U, \varphi(\delta) > \mu(V \cap U) - \varepsilon$$

Consider the open set $W = V \setminus \text{supp } \delta$.



Choose $0 \leq g \leq 1, \text{ supp } g \subset W, \varphi(g) > \mu(W) - \varepsilon$.

Then

$$\begin{aligned} \mu(V \cap U) + \mu^*(V \cap U^c) &\leq \mu(V \cap U) + \mu(W) \\ &< \varphi(\delta) + \varphi(g) + 2\varepsilon = \varphi(\delta + g) + 2\varepsilon \\ &\leq \mu(V) + 2\varepsilon, \end{aligned}$$

since $0 \leq \delta + g \leq 1, \text{ supp } (\delta + g) \subset V$.

As $\varepsilon > 0$ was arbitrary, we get the required inequality.

For general A , take $V \supset A$, V open. Then

$$\mu(V) \geq \mu(V \cap U) + \mu^*(V \cap U^c) \geq \mu^*(A \cap U) + \mu^*(A \cap U^c)$$

Taking the infimum over all V we get

$$\mu^*(A) \geq \mu^*(A \cap U) + \mu^*(A \cap U^c).$$