

Step 3 If $K \subset X$ is compact, then $\mu(K) < \infty$ and $\mu(K) = \inf \{ \varphi(f) : \pi_K \leq f \}$ ①

Also, if $0 \leq f \leq \pi_K$, then $\varphi(f) \leq \mu(K)$.

Assume $\pi_K \leq f$. Fix $\varepsilon > 0$ and consider the open set $U = \{x : f(x) > 1 - \varepsilon\}$.

Assume $0 \leq g \leq 1$, $\text{supp } g \subset U$. Then

$$g \leq \frac{f}{1-\varepsilon}, \text{ hence } \varphi(g) \leq \frac{\varphi(f)}{1-\varepsilon}.$$

Taking the supremum over all such g , we get $\mu(U) \leq \frac{\varphi(f)}{1-\varepsilon}$. Hence

$$\mu(K) \leq \mu(U) \leq \frac{\varphi(f)}{1-\varepsilon}.$$

As $\varepsilon > 0$ was arbitrary, we get $\mu(K) \leq \varphi(f)$. In particular, $\mu(K) < \infty$. (Note that $f \in C_c(X)$ s.t. $\pi_K \leq f$ exists by Urysohn's lemma.) We see also that

$$\mu(K) \leq \inf \{ \varphi(f) : \pi_K \leq f \}.$$

In order to prove the equality, fix $\varepsilon > 0$ and choose open U s.t. $K \subset U$, $\mu(K) > \mu(U) - \varepsilon$.

By Urysohn's lemma we can find $f \in C_c(X)$ s.t. $0 \leq f \leq 1$, $f \equiv 1$ on K (so $\pi_K \leq f$) and $\text{supp } f \subset U$. Then

$$\psi(\delta) \leq \mu(U) < \mu(K) + \epsilon.$$

Therefore

$$\inf \{ \psi(\delta) : \pi_K \leq \delta \} < \mu(K) + \epsilon.$$

As $\epsilon > 0$ was arbitrary, we conclude that

$$\mu(K) = \inf \{ \psi(\delta) : \pi_K \leq \delta \}.$$

Finally, assume $0 \leq \delta \leq \pi_K$.

Then for every open $U \supset K$ we have $\text{supp } \delta \subset U$, so $\psi(\delta) \leq \mu(U)$. Hence

$$\psi(\delta) \leq \inf_{\substack{K \subset U \\ U \text{ open}}} \mu(U) = \mu(K).$$

Step 4 $\psi(\delta) = \int_X \delta d\mu \quad \forall \delta \in C_c(X).$

It suffices to consider $0 \leq \delta \leq 1$, as such functions span $C_c(X)$.

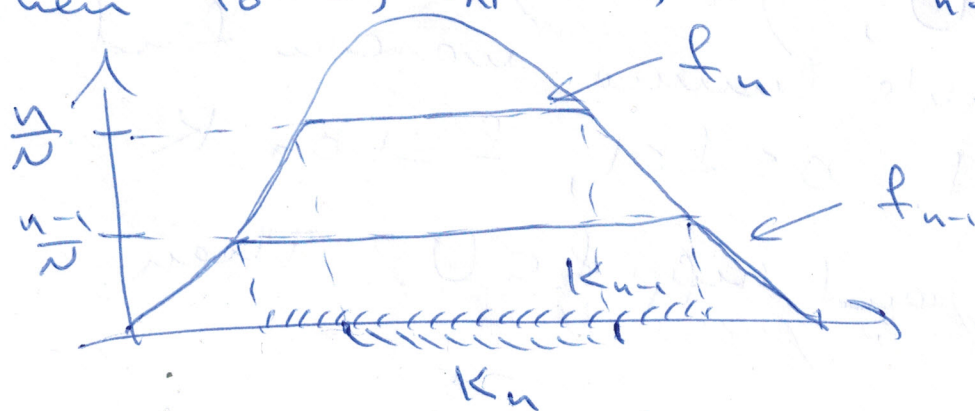
Consider $K = \text{supp } \delta$ and define for some N

$$K_n = \{ x \in K : \delta(x) \geq \frac{n}{N} \} \quad (n=0, \dots, N),$$

$$f_n(x) = \min \{ \delta(x), \frac{n}{N} \} \quad (n=0, \dots, N),$$

$$g_n(x) = f_n(x) - f_{n-1}(x), \quad (n=1, \dots, N).$$

Then $f_0 = 0$, $f_N = \delta$, so $\delta = \sum_{n=1}^N g_n$.



we have (2)

$$\frac{1}{N} \pi_{K_n} \leq g_n \leq \frac{1}{N} \pi_{K_{n-1}} \quad (n=1, \dots, N)$$

By integrating we get

$$\frac{1}{N} \mu(K_n) \leq \int_X g_n d\mu \leq \frac{1}{N} \mu(K_{n-1}).$$

On the other hand by Step 3 we have

$$\frac{1}{N} \mu(K_n) \leq \psi(g_n) \leq \frac{1}{N} \mu(K_{n-1}).$$

It follows that both $\int_X f d\mu$ and $\psi(f)$ lie in the interval

$$\left[\frac{1}{N} \sum_{n=1}^N \mu(K_n), \frac{1}{N} \sum_{n=1}^N \mu(K_{n-1}) \right].$$

The length ~~of~~ of this interval is

$$\frac{1}{N} (\mu(K_0) - \mu(K_N)) \leq \frac{1}{N} \mu(K_0) = \frac{\mu(K)}{N}.$$

Therefore

$$\left| \int_X f d\mu - \psi(f) \right| \leq \frac{\mu(K)}{N}.$$

Letting $N \rightarrow \infty$ we conclude that

$$\psi(f) = \int_X f d\mu.$$

Step 5 If X is separable, then μ is uniquely determined by ψ .

Assume μ is a Borel measure s.t.

$\mu(K) < \infty$ $\forall$ compact $K \subset X$ and

$$\psi(f) = \int_X f d\mu \quad \forall f \in C_c(X).$$

Recall from the lecture on regularity of measures that then μ is regular (as X is separable, locally compact).

In particular,

$$\mu(A) = \inf_{\substack{A \subset U \\ U \text{ open}}} \mu(U) \quad \forall A \in \mathcal{B}(X).$$

Therefore it suffices to show that $\mu(U)$ is determined by φ .

If $\bar{U}$ is compact, then $f_n \uparrow \chi_U$, where $f_n(x) = \frac{\text{nd}(x, U^c)}{1 + \text{nd}(x, U^c)}$. Hence

$$\mu(U) = \lim_{n \rightarrow \infty} \int_X f_n d\mu = \lim_{n \rightarrow \infty} \varphi(f_n).$$

For general U , we can find open U_n s.t. $\bar{U}_n$ are compact and $U_n \uparrow U$ (again, recall the lecture on regularity). Then $\mu(U_n) \uparrow \mu(U)$, so $\mu(U)$ is determined by φ . □

Remark

Without separability the uniqueness is not always true. It can be checked that the measure we constructed is the proof has the following properties:

- 1) $\mu(K) < \infty$ if compact $K \subset X$;

2) μ is outer regular ($\mu(A) = \inf_{\substack{A \subset U \\ U \text{ open}}} \mu(U)$) ;

3) μ is inner regular on open sets ;

$$\mu(U) = \sup_{\substack{K \subset U \\ K \text{ compact}}} \mu(K) \quad \text{if open } U$$

(This is where we need the full strength of Step 3)

Such measures are called Radon measures

It can be shown that the uniqueness holds within the ~~class~~ class of Radon measures.

Dual of $C(X)$

As an application of the Riesz-Markov theorem we will describe $C(X)^*$ in terms of measures for compact metric spaces ~~metric~~ (X, d) .

Denote by $M(X)$ the space of complex Borel measures on X .

For every $\nu \in M(X)$ we want to make sense of $\int_X f d\nu$ for $f \in C(X)$.

It is enough to consider finite signed measures, as then we can define

$$\int_X f d\nu = \int_X f d(\operatorname{Re} \nu) + i \int_X f d(\operatorname{Im} \nu).$$

So assume ν is a finite signed measure.

Then $\nu = \mu_1 - \mu_2$ for positive measures

and we define

$$\int_X f d\nu = \int_X f d\mu_1 - \int_X f d\mu_2.$$

This is well-defined, since if

$$\nu = \mu_1 - \mu_2 = \omega_1 - \omega_2,$$

then $\mu_1 + \omega_2 = \mu_2 + \omega_1$ and

$$\int_X f d\mu_1 + \int_X f d\omega_2 = \int_X f d\mu_2 + \int_X f d\omega_1.$$

Thus, every $\nu \in M(X)$ defines a linear functional

$$\varphi_\nu : C(X) \rightarrow \mathbb{C} \quad \text{by } \varphi_\nu(f) = \int_X f d\nu,$$

and the map ~~map~~ $\nu \mapsto \varphi_\nu$ is linear.

Lemma

If $\nu \in M(X)$ and $d\nu = g d|\nu|$ is its polar decomposition, then

$$\int_X f d\nu = \int_X f g d|\nu| \quad \forall f \in C(X).$$

Proof

More generally, let us show that

if μ is a finite Borel measure, $g \in L^1(X, d\mu)$, and $d\nu = g d\mu$, then

$$\int_X f d\nu = \int_X f g d\mu \quad \forall f \in C(X).$$

It suffices to consider $g \geq 0$.

Then ν is positive, so $\int_X f d\nu$ has the usual meaning.

~~Lemma~~ We will then prove an even more general statement! (9)

$$\int_X f d\nu = \int_X f g d\mu \quad \forall f \in L^\infty(X, \mu).$$

As simple functions are dense in $L^\infty(X, \mu)$, it suffices to consider ~~simple~~ simple f , hence it suffices to consider $f = \mathbb{1}_A$. Then

$$\int_X \mathbb{1}_A d\nu = \nu(A) = \int_X \mathbb{1}_A g d\mu = \int_X \mathbb{1}_A g d\mu$$

by the definition
of $d\nu = g d\mu$

□

Lemma

For every $\nu \in M(X)$, the linear functional φ_ν is bounded and $\|\varphi_\nu\| = |\nu|(X)$.

Recall that the norm on $C(X)$ is $\|f\| = \sup_{x \in X} |f(x)|$.

Proof

we have, with $d\nu = g d|\nu|$ being the polar decomposition,

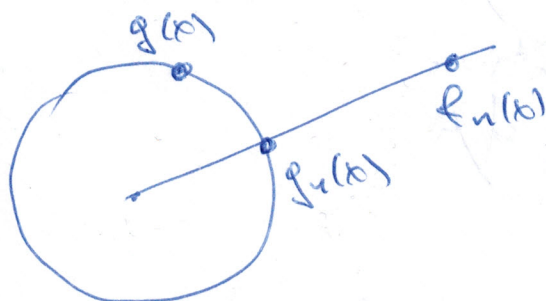
$$\begin{aligned} |\varphi_\nu(f)| &= \left| \int_X f d\nu \right| = \left| \int_X f g d|\nu| \right| \\ &\leq \int_X |f| d|\nu| \leq \|f\| \int_X 1 d|\nu| \\ &\leq \|f\| |\nu|(X). \end{aligned}$$

Therefore $\|\varphi_\nu\| \leq |\nu|(X)$.

For the opposite inequality, recall that $|v|$ is regular and $C(X)$ is dense in $L^1(X, d|v|)$. Hence we can find $f_n \in C(X)$ s.t. $f_n \rightarrow g$ in $L^1(X, d|v|)$. Define

$$g_n(x) = \begin{cases} f_n(x), & \text{if } |f_n(x)| \leq 1, \\ \frac{f_n(x)}{|f_n(x)|}, & \text{if } |f_n(x)| > 1. \end{cases}$$

Then $\|g_n\| \leq 1$ and $|g - g_n| \leq |g - f_n|$.



Therefore $\|g - g_n\| \leq \|g - f_n\| \xrightarrow{n \rightarrow \infty} 0$.

We have

$$\varphi_v(\bar{g}_n) = \int_X \bar{g}_n g d|v| \rightarrow \int_X |d|v| = |v|(X),$$

since

$$\left| \int_X (\bar{g}_n g - 1) d|v| \right| \leq \int_X |\bar{g}_n g - 1| d|v|$$

$$= \int_X |\bar{g}_n - \bar{g}| d|v| = \|g_n - g\| \xrightarrow{n \rightarrow \infty} 0$$

as $|g(x)| = 1$

Therefore $\|\varphi_v\| \geq \sup_n |\varphi_v(\bar{g}_n)| \geq |v|(X)$.