

Lemma

Assume X is a compact metric space.

1) If $\varphi: C(X) \rightarrow \mathbb{C}$ is a positive linear functional, then φ is bounded and $\|\varphi\| = \varphi(1)$.

2) Every bounded linear functional on $C(X)$ is a linear combination of positive linear functionals.

Proof

1) Define $(\cdot, \cdot): C(X) \times C(X) \rightarrow \mathbb{C}$ by
 $(f, g) = \varphi(f \bar{g})$.

This is a pre-inner product on $C(X)$ by positivity of φ . Hence, by the Cauchy-Schwarz inequality we have

$$|(f, g)| \leq (f, f)^{1/2} (g, g)^{1/2}.$$

Applying this to $g = 1$ we get

$$|\varphi(f)| \leq \varphi(|f|^2)^{1/2} \varphi(1)^{1/2}$$

As $|f|^2 \leq \|f\|^2 \cdot 1$, we have

$$\varphi(|f|^2) \leq \|f\|^2 \varphi(1).$$

Therefore

$$|\varphi(f)| \leq \|f\| \varphi(1)^{1/2}.$$

This shows that φ is bounded and

$$\| \varphi \| \leq \varphi(1).$$

On the other hand, $\| \varphi \| \geq \varphi(1)$, since $\| 1 \| = 1$. Hence $\| \varphi \| = \varphi(1)$.

2) For $\varphi \in C(X)^*$ define linear functionals φ_1, φ_2 by

$$\varphi_1(f) = \frac{1}{2} (\varphi(f) + \overline{\varphi(\bar{f})}),$$

$$\varphi_2(f) = \frac{1}{2i} (\varphi(f) - \overline{\varphi(\bar{f})}).$$

Then $\varphi_1, \varphi_2 \in C(X)^*$, $\varphi = \varphi_1 + i\varphi_2$ and

$$\varphi_j(\bar{f}) = \overline{\varphi_j(f)} \text{ for } f \in C(X) \text{ and } j=1,2.$$

Therefore it suffices to show that if

$\varphi \in C(X)^*$ satisfies $\varphi(\bar{f}) = \overline{\varphi(f)}$, then

φ is a linear combination of positive linear functionals. This is shown by

an analogue of Jordan's decomposition as follows:

Define, for $f \in C(X)$, $f \geq 0$,

$$\varphi_+(f) \stackrel{\text{def}}{=} \sup \{ \varphi(g) : 0 \leq g \leq f \}.$$

Note that $\varphi_+(f)$ is finite, since if

$0 \leq g \leq f$, then $|\varphi(g)| \leq \| \varphi \| \| g \| \leq \| \varphi \| \| f \|$.

Since we can take $g \geq 0$, we have $\varphi_+(f) \geq 0$.

Obviously, $\varphi_+(t f) = t \varphi_+(f)$ ②

for all $f \in C(X)$, $f \geq 0$, and $t \geq 0$.

Let us show next that if $f_1, f_2 \geq 0$, then

$$\varphi_+(f_1 + f_2) = \varphi_+(f_1) + \varphi_+(f_2).$$

If $0 \leq g_1 \leq f_1$, $0 \leq g_2 \leq f_2$, then $0 \leq g_1 + g_2 \leq f_1 + f_2$.

This implies that

$$\varphi_+(f_1 + f_2) \geq \varphi_+(f_1) + \varphi_+(f_2).$$

To prove the opposite inequality, take $0 \leq g \leq f_1 + f_2$. Define

$$g_1 = \min\{g, f_1\} \quad \text{and} \quad g_2 = g - g_1 \geq 0$$

We have $0 \leq g_1 \leq f_1$. Also, for every $x \in X$,

$$g_2(x) = g(x) - g_1(x) = \begin{cases} 0, & \text{if } g(x) \leq f_1(x) \\ g(x) - f_1(x), & \text{if } g(x) > f_1(x) \end{cases}$$

$$\leq \underbrace{(f_1(x) + f_2(x)) - f_1(x)}_{= f_2(x)} = f_2(x).$$

Then

$$\varphi(g) = \varphi(g_1) + \varphi(g_2) \leq \varphi_+(f_1) + \varphi_+(f_2).$$

$$\text{Hence } \varphi_+(f_1 + f_2) \leq \varphi_+(f_1) + \varphi_+(f_2).$$

This proves that $\varphi_+(f_1 + f_2) = \varphi_+(f_1) + \varphi_+(f_2)$.

We can now extend φ_+ to a positive linear functional on $C(X)$ in the same way, as we extended integrals from positive

functions:

First we extend φ_+ to real-valued functions by

$$\varphi_+(f_1 - f_2) \stackrel{\text{def}}{=} \varphi_+(f_1) - \varphi_+(f_2)$$

for $f_1, f_2 \geq 0$. This is well-defined, since

$$\text{if } f_1 - f_2 = g_1 - g_2 \quad (f_1, f_2, g_1, g_2 \geq 0),$$

then $f_1 + g_2 = f_2 + g_1$ and hence

$$\varphi_+(f_1) + \varphi_+(g_2) = \varphi_+(f_2) + \varphi_+(g_1).$$

Then we extend φ_+ to complex-valued functions by

$$\varphi_+(f) \stackrel{\text{def}}{=} \varphi_+(\operatorname{Re} f) + i\varphi_+(\operatorname{Im} f).$$

Finally, consider $\varphi_- \stackrel{\text{def}}{=} \varphi_+ - \varphi$.

For $f \geq 0$ we have, by definition,

$$\varphi_+(f) \geq \varphi(f), \text{ hence } \varphi_- \text{ is positive.}$$

We also have $\varphi = \varphi_+ - \varphi_-$. □

Theorem

Assume X is a compact metric space.

Consider the space $M(X)$ of complex Borel

measures on X . Then the map

$$M(X) \rightarrow C(X)^*, \quad \nu \mapsto \varphi_\nu, \quad \varphi_\nu(f) \stackrel{\text{def}}{=} \int_X f d\nu,$$

is a linear isomorphism.

Proof

③

We have already shown that $\|\varphi_v\| = \|v\|_1(X)$.
In particular, $\varphi_v = 0$ if and only if $v = 0$.

This shows that the map

$$M(X) \rightarrow C(X)^*, v \mapsto \varphi_v,$$

is injective. The image of this map contains all positive linear functionals by the Riesz-Markov theorem. Since such functionals span the entire space $C(X)^*$ by the previous lemma, it follows that the ~~image~~ map $M(X) \rightarrow C(X)^*$ is surjective. $\square$

Corollary

The vector space $M(X)$ is a Banach space with norm $\|v\| = \|v\|_1(X)$.

Remark

More generally, similar arguments show that we have a linear isomorphism $M_{\text{reg}}(X) \cong C_0(X)^*$ for every locally compact metric space X , where $M_{\text{reg}}(X)$ denotes the space of regular complex Borel measures on X , where v is called regular when $|v|$ is regular (equivalently, $(\operatorname{Re} v)_+$, $(\operatorname{Im} v)_+$ are regular).

Product-measures

Assume $(X_1, \mathcal{B}_1, \mu_1), (X_2, \mathcal{B}_2, \mu_2)$ are measure spaces. Consider $X = X_1 \times X_2$.

Denote by $\mathcal{B}_1 \times \mathcal{B}_2$ the σ -algebra generated by all sets of the form $A_1 \times A_2$ with $A_1 \in \mathcal{B}_1, A_2 \in \mathcal{B}_2$.

Can we define a measure on $(X_1 \times X_2, \mathcal{B}_1 \times \mathcal{B}_2)$ s.t.

$$\mu(A_1 \times A_2) = \mu_1(A_1) \mu_2(A_2)?$$

We start with the following technical result.

Theorem

Assume $(X_1, \mathcal{B}_1, \mu_1), (X_2, \mathcal{B}_2, \mu_2)$ are σ -finite measure spaces, $f: X_1 \times X_2 \rightarrow [0, +\infty]$

is a $(\mathcal{B}_1 \times \mathcal{B}_2)$ -measurable function. Then

1) the function $X_1 \rightarrow [0, +\infty], x_1 \mapsto f(x_1, x_2)$,
is $\mathcal{B}_1$ -measurable $\forall x_2 \in X_2$;

2) the function $X_2 \rightarrow [0, +\infty], x_2 \mapsto f(x_1, x_2)$,
is $\mathcal{B}_2$ -measurable $\forall x_1 \in X_1$;

3) the function $X_1 \rightarrow [0, +\infty], x_1 \mapsto \int_{X_2} f(x_1, x_2) d\mu_2(x_2)$,
is $\mathcal{B}_1$ -measurable;

4) the function $X_2 \rightarrow [0, +\infty], x_2 \mapsto \int_{X_1} f(x_1, x_2) d\mu_1(x_1)$,
is $\mathcal{B}_2$ -measurable;

5) $\int_{X_1} \left(\int_{X_2} f(x_1, x_2) d\mu_2(x_2) \right) d\mu_1(x_1) = \int_{X_2} \left(\int_{X_1} f(x_1, x_2) d\mu_1(x_1) \right) d\mu_2(x_2)$

Proof

(4)

Assume first that $\mu_1(X_1), \mu_2(X_2) < \infty$.

The theorem is immediate for

$$f(x_1, x_2) = \prod_{A_1} \prod_{A_2} f(x_1, x_2),$$

that is, for $f = \prod_{A_1 \times A_2}$, where $A_1 \in \mathcal{B}_1, A_2 \in \mathcal{B}_2$.

Consider the collection $\mathcal{D}$ of sets $A \in \mathcal{B}_1 \times \mathcal{B}_2$

s.t. the theorem is true for $f = \prod_A$.

Using the monotone convergence theorem

and the equality $\prod_{A^c} = 1 - \prod_A$ we

can easily see that $\mathcal{D}$ is a Dynkin system.

Since $\mathcal{D}$ contains the sets $A_1 \times A_2$ ($A_j \in \mathcal{B}_j$)

and the collection of such sets is closed

under finite intersections and generates

$\mathcal{B}_1 \times \mathcal{B}_2$ as a σ -algebra, by Dynkin's theorem

we conclude that $\mathcal{D} = \mathcal{B}_1 \times \mathcal{B}_2$.

It follows then that the theorem is

true for positive simple functions, and then, by the monotone convergence theorem, for all positive measurable functions.

Consider now the general σ -finite case.

Then there exist $X_{jn} \subset X_j$ ($j=1,2, n \in \mathbb{N}$)

s.t. $X_{jn} \uparrow X_j$ as $n \rightarrow \infty$ and $\mu_j(X_{jn}) < \infty$.

Given any measurable $f: X_1 \times X_2 \rightarrow [0, \infty]$, the theorem is true for $f \prod_{X_{1n} \times X_{2n}}$

by the first part of the proof.

As $f \uparrow \prod_{X_1 \times X_2} \uparrow f$ pointwise, by

the monotone convergence theorem, the result is true for f . □