

Theorem

①

Assume $(X_1, \mathcal{B}_1, \mu_1), (X_2, \mathcal{B}_2, \mu_2)$ are σ -finite measure spaces. Then there is a unique measure $\mu = \mu_1 \times \mu_2$ on $(X_1 \times X_2, \mathcal{B}_1 \times \mathcal{B}_2)$

s.t. $\mu(A_1 \times A_2) = \mu_1(A_1) \mu_2(A_2) \quad \forall A_1 \in \mathcal{B}_1, \forall A_2 \in \mathcal{B}_2.$

Proof

Uniqueness follows from Dynkin's theorem.

For the existence, define

$$\begin{aligned} \mu(A) &\stackrel{\text{def}}{=} \sum_{X_1} \left(\sum_{X_2} \mathbb{1}_A(x_1, x_2) d\mu_2(x_2) \right) d\mu_1(x_1) \\ &= \sum_{X_2} \left(\sum_{X_1} \mathbb{1}_A(x_1, x_2) d\mu_1(x_1) \right) d\mu_2(x_2). \end{aligned}$$

By using the monotone convergence theorem we see that this is indeed a measure, since if $A = \bigcup_{n=1}^{\infty} A_n, A_n \cap A_m = \emptyset$

for $n \neq m$, then $\sum_{k=1}^n \mathbb{1}_{A_k} \uparrow \mathbb{1}_A$ as $n \rightarrow \infty$. $\square$

Theorem (Fubini-Tonelli)

Assume $(X_1, \mathcal{B}_1, \mu_1), (X_2, \mathcal{B}_2, \mu_2)$ are σ -finite measure spaces. Consider the measure

$\mu = \mu_1 \times \mu_2$ on $(X_1 \times X_2, \mathcal{B}_1 \times \mathcal{B}_2)$.

1) Tonelli: if $f: X_1 \times X_2 \rightarrow \{0, +\infty\}$ is $(\mathcal{B}_1 \times \mathcal{B}_2)$ -measurable, then

$$\begin{aligned} \int_{X_1 \times X_2} f d\mu &= \sum_{X_1} \left(\sum_{X_2} f(x_1, x_2) d\mu_2(x_2) \right) d\mu_1(x_1) \\ &= \sum_{X_2} \left(\sum_{X_1} f(x_1, x_2) d\mu_1(x_1) \right) d\mu_2(x_2). \end{aligned}$$

2) Fubini: if $f: X_1 \times X_2 \rightarrow \mathbb{R}$ is $(\mathcal{B}_1 \times \mathcal{B}_2)$ -measurable and any of the integrals

$$\int_{X_1} |f| d\mu, \int_{X_1} \left(\int_{X_2} |f(x_1, x_2)| d\mu_2(x_2) \right) d\mu_1(x_1),$$

$$\int_{X_2} \left(\int_{X_1} |f(x_1, x_2)| d\mu_1(x_1) \right) d\mu_2(x_2)$$

is finite, then f is μ -integrable and

$$\begin{aligned} \int_X f d\mu &= \int_{X_1} \left(\int_{X_2} f(x_1, x_2) d\mu_2(x_2) \right) d\mu_1(x_1) \\ &= \int_{X_2} \left(\int_{X_1} f(x_1, x_2) d\mu_1(x_1) \right) d\mu_2(x_2). \end{aligned}$$

Warning

In Fubini's theorem it is ~~usually~~ claimed that $\int_{X_2} f(x_1, x_2) d\mu_2(x_2)$ is well-defined only for μ_1 -a.e. x_1 , similarly $\int_{X_1} f(x_1, x_2) d\mu_1(x_1)$ is well-defined only for μ_2 -a.e. x_2 .

Proof

The result is true for $f = \mathbb{1}_A$ ($A \in \mathcal{B}_1 \times \mathcal{B}_2$)

by the definition of $\mu = \mu_1 \times \mu_2$.

By Tonelli's theorem follows from this by the usual ~~approximation~~ passage to positive simple functions and then all positive measurable functions.

In 2), by Tonelli's theorem, the three integrals of $|f|$ are always equal.

So if any of them is finite, then $\int_{X_1 \times X_2} |f| d\mu < \infty$.

The final two equalities are true for $f \geq 0$ by Tonelli's theorem. The general case reduces to this, since every integrable f is a linear combination of positive integrable functions. Note also that as $\int_{X_1 \times X_2} |f| d\mu = \int_{X_1} \left(\int_{X_2} |f(x_1, x_2)| d\mu_2(x_2) \right) d\mu_1(x_1)$, we have $\int_{X_2} |f(x_1, x_2)| d\mu_2(x_2) < \infty$ for μ_1 -a.e. x_1 ,

so $\int_{X_2} f(x_1, x_2) d\mu_2(x_2) \in \mathbb{C}$ is well-defined for μ_1 -a.e. x_1 . □

Corollary

For $A \in \mathcal{B}_1 \times \mathcal{B}_2$ and $x_1 \in X_1$, define

$$A(x_1) \stackrel{\text{def}}{=} \{x_2 \in X_2 : (x_1, x_2) \in A\}.$$

Then $(\mu_1 \times \mu_2)(A) = 0$ if and only if $\mu_2(A(x_1)) = 0$ for μ_1 -a.e. $x_1 \in X_1$.

Proof

We have

$$\mathbb{1}_A(x_1, x_2) = \mathbb{1}_{A(x_1)}(x_2).$$

$$\begin{aligned} \text{Hence} \\ \mu(A) &= \int_{X_1} \left(\int_{X_2} \mathbb{1}_A(x_1, x_2) d\mu_2(x_2) \right) d\mu_1(x_1) \\ &= \int_{X_1} \mu_2(A(x_1)) d\mu_1(x_1). \end{aligned}$$

It follows that $\mu(A) = 0$ if and only if $\mu_2(A(x_1)) = 0$ for μ_1 -a.e. $x_1 \in X_1$. □

Lebesgue measure on $\mathbb{R}^n$

We can now construct the Lebesgue measure λ_n on $\mathbb{R}^n$ in two ways:

- 1) Define a positive linear functional φ on $C_c(\mathbb{R}^n)$ by

$$\varphi(f) = \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} f(x_1, \dots, x_n) dx_1 \dots dx_n.$$

By the Riesz-Markow theorem this functional gives a Borel measure on $\mathbb{R}^n$. By completing we get the σ -algebra of Lebesgue measurable sets and the measure λ_n on it.

- 2) Consider the Lebesgue measure λ on $B(\mathbb{R})$. Then we can consider $\underbrace{\lambda \times \dots \times \lambda}_n$ on $\underbrace{B(\mathbb{R}) \times \dots \times B(\mathbb{R})}_n = B(\mathbb{R}^n)$,

complete and get the σ -algebra of Lebesgue measurable sets and the measure λ_n on it.

That the results of these two constructions are the same can be checked by comparing the measures of $\{a_1, b_1\} \times \dots \times \{a_n, b_n\}$.

Theorem

(8)

Assume $\Omega_1, \Omega_2 \subset \mathbb{R}^n$ are open sets,
 $F: \Omega_1 \rightarrow \Omega_2$ is a C^1 -diffeomorphism.
Consider the restrictions μ_1, μ_2 of λ_n
to Ω_1 and Ω_2 . Then F is Lebesgue
measurable, $F_*\mu_1 \ll \mu_2$ and

$$\frac{d(F_*\mu_1)}{d\mu_2} = |\det T_F|^{-1} = |\det T_F|^{-1} \circ F^{-1},$$

where T_F is the Jacobian ~~matrix~~
matrix of F , $T_F(x) = \left(\frac{\partial F_i}{\partial x_j}(x) \right)_{i,j=1}^n$.

Proof

From calculus we (are supposed to) know that

$$\int_{\Omega_1} f \circ F |\det T_F| d\lambda_n = \int_{\Omega_2} f d\lambda_n \quad (*)$$

for all $f \in C_c(\Omega_2)$.

By replacing f by $f |\det T_F|^{-1}$
and using that $|\det T_F|^{-1} = |\det T_F|^{-1} \circ F^{-1}$,
we can rewrite (*) as

$$\int_{\Omega_1} f \circ F d\mu_1 = \int_{\Omega_2} f |\det T_F|^{-1} d\mu_2.$$

Consider the measure ν s.t. $d\nu = |\det T_F|^{-1} d\mu_2$.

$$\text{As } \int_{\Omega_1} f \circ F d\mu_1 = \int_{\Omega_2} f d(F_*\mu_1),$$

we see that $F_*\mu_1$ and ν define

the same linear functional on $C_c(\Omega_2)$.
Hence, by the uniqueness part of
the Riesz-Markow theorem, we have

$F_*\mu_1 = \nu$ on $\mathcal{B}(\Omega_2)$. Therefore if
we considered only Borel σ -algebras, we
could already conclude that

$$\frac{d(F_*\mu_1)}{d\mu_2} = \frac{d\nu}{d\mu_2} = |\det T_F^{-1}|.$$

But the difference between Borel and
Lebesgue measurable sets is not large:
every Lebesgue measurable set $A \subset \Omega_2$
has the form $A = B \cup C$, where $B \in \mathcal{B}(\Omega_2)$,
 $C \subset \tilde{C}$, $\tilde{C} \in \mathcal{B}(\Omega_2)$ and $\mu_2(\tilde{C}) = 0$.

Then $F^{-1}(C) \subset F^{-1}(\tilde{C})$ and

$$\mu_1(F^{-1}(\tilde{C})) = (F_*\mu_1)(\tilde{C}) = \nu(\tilde{C}) = 0.$$

Hence $F^{-1}(C)$ is Lebesgue measurable, and

$$\text{hence so is } F^{-1}(A) = F^{-1}(B) \cup F^{-1}(C).$$

Therefore F is indeed Lebesgue measurable
and using again that every Lebesgue measurable
set of measure zero is contained in a Borel
set of measure zero we can check that
 $F_*\mu_1 \ll \mu_2$ and the Radon-Nikodym
derivative remains the same,

$$\frac{dF_*\mu_1}{d\mu_2} = |\det T_F^{-1}|.$$