

Fourier transform

①

write $L^1(\mathbb{R}^n)$ for $L^1(\mathbb{R}^n, d\lambda_n)$.

The Fourier transform of a function $f \in L^1(\mathbb{R}^n)$ is the function $\hat{f}$ on $\mathbb{R}^n$ defined by

$$\hat{f}(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} f(y) e^{-i\langle x, y \rangle} dy,$$

where $\langle x, y \rangle = x_1 y_1 + \dots + x_n y_n$.

More generally, given a finite Borel measure μ , its Fourier transform is the function $\hat{\mu}$ on $\mathbb{R}^n$ defined by

$$\hat{\mu}(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{-i\langle x, y \rangle} d\mu(y)$$

we can also define $\hat{\mu}$ for complex Borel measures.

Warning

There are many different conventions for the Fourier transform: instead of $\frac{1}{(2\pi)^n}$ one also uses 1, $\frac{1}{(2\pi)^{n/2}}$; instead of $e^{-i\langle x, y \rangle}$ one also uses $e^{i\langle x, y \rangle}$, $e^{\pm 2\pi i \langle x, y \rangle}$.

Note that if μ_f for $f \in L^1(\mathbb{R}^n)$ is defined by $d\mu_f = f d\lambda_n$, then

$$\hat{\mu}_f = \hat{f}.$$

Lemma

If μ is a complex Borel measure on $\mathbb{R}^n$, then $\hat{\mu}$ is a bounded continuous function on $\mathbb{R}^n$, and $|\hat{\mu}(x)| \leq \frac{\|\mu\|(\mathbb{R}^n)}{(2\pi)^n}$.

In particular, if $f \in L^1(\mathbb{R}^n)$, then $\hat{f}$ is bounded and continuous,
 $|\hat{f}(x)| \leq \frac{\|f\|_1}{(2\pi)^n}, \forall x.$

Proof

$$\text{We have } |\hat{f}(x)| = \left| \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{-i\langle x, y \rangle} d\mu(y) \right| \\ \leq \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} 1 d\mu(y) = \frac{1}{(2\pi)^n} \mu(\mathbb{R}^n).$$

Continuity follows from the dominated convergence theorem. $\square$

Some properties:

1. If $f_t(x) = f(x-t)$, then

$$\hat{f}_t(y) = e^{-i\langle t, y \rangle} \hat{f}(y).$$

2. If $e_t(x) = e^{i\langle t, x \rangle}$, then

$$\widehat{e_t f}(y) = \hat{f}(y-t).$$

3. If $T \in GL_n(\mathbb{R})$ (invertible $n \times n$ matrices),

then

$$\widehat{f \circ T} = |\det T|^{-1} \hat{f} \circ (T^+)^{-1}$$

Proof

$$\begin{aligned} \widehat{f \circ T}(x) &= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} f(Ty) e^{-i\langle x, y \rangle} dy \\ &= \frac{1}{|\det T|} \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} f(y) e^{-i\langle x, T^{-1}y \rangle} dy \\ &= \frac{1}{|\det T|} \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} f(y) e^{-i\langle (T^+)^{-1}x, y \rangle} dy \\ &= \frac{1}{|\det T|} \hat{f}((T^+)^{-1}x). \end{aligned} \quad \square$$

4. $\widehat{f}(x) = \widehat{f}(-x)$.

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Important example

If $f(x) = e^{-\frac{|x|^2}{2}}$ ($|x| = (x, x)^{1/2}$),

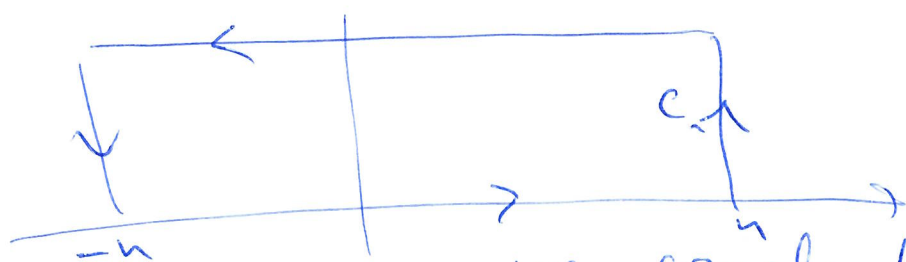
then $\widehat{f}(x) = \frac{1}{(2\pi)^{1/2}} e^{-\frac{|x|^2}{2}}$.

Proof we need to show that

$$\frac{1}{2\pi} \int_{\mathbb{R}} e^{-\frac{y^2}{2} - ixy} dy = \frac{e^{-x^2/2}}{(2\pi)^{1/2}}$$

we have $e^{-\frac{y^2}{2} - ixy} = e^{-\frac{1}{2}(y+ix)^2 - \frac{x^2}{2}}$,
so we need to check that

$$\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-\frac{1}{2}(y+ix)^2} dy = 1$$



$$\oint_{C_n} e^{-\frac{z^2}{2}} dz = 0$$

Letting $n \rightarrow +\infty$, we conclude that

$$\int_{\mathbb{R}} e^{-\frac{1}{2}(y+ix)^2} dy = \int_{\mathbb{R}} e^{-\frac{y^2}{2}} dy$$

It remains to show that

$$\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-\frac{y^2}{2}} dy = 1$$

Let $a = \int_0^{\infty} e^{-\frac{y^2}{2}} dy$, so $\int_{\mathbb{R}} e^{-\frac{y^2}{2}} dy = 2a$.

Then $a^2 = \int_0^{\infty} \int_0^{\infty} e^{-\frac{x^2+y^2}{2}} dx dy$

Passing to spherical coordinates we get

$$a^2 = \int_0^{\infty} \int_0^{2\pi} \int_0^{\frac{\pi}{2}} e^{-\frac{r^2}{2}} r \sin \theta dr d\theta d\phi = \frac{\pi}{2} \int_0^{\infty} e^{-\frac{r^2}{2}} r dr = \frac{\pi}{2}$$

Therefore $2a = \sqrt{2\pi}$. □

More generally, if $f(x) = e^{-\frac{c|x|^2}{2}}$, then

$$\hat{f}(x) = \frac{1}{(2\pi c)^{\frac{n}{2}}} e^{-\frac{|x|^2}{2c}} \quad \forall c > 0.$$

This follows from property 3.

For functions f, g on $\mathbb{R}^n$, their convolution is the function $f \times g$ defined by

$$(f \times g)(x) = \int_{\mathbb{R}^n} f(y) g(x-y) dy = \int_{\mathbb{R}^n} f(x-y) g(y) dy$$

When is this well-defined?

Lemma

If $f, g \in L^1(\mathbb{R}^n)$, then the function
 $y \mapsto f(y) g(x-y)$ is integrable for (λ_n) -a.e. x ,
 $f \times g \in L^1(\mathbb{R}^n)$ and $\|f \times g\|_1 \leq \|f\|_1 \|g\|_1$.

Proof

For everything except the last inequality we may assume that $f, g \geq 0$.

Consider $F: \mathbb{R}^n \times \mathbb{R}^n \rightarrow [0, +\infty)$,

$$F(x, y) = f(x-y) g(y).$$

Then

$$\begin{aligned} \sum_{\mathbb{R}^n} \left(\sum_{\mathbb{R}^n} F(x, y) dx \right) dy &= \sum_{\mathbb{R}^n} \left(\sum_{\mathbb{R}^n} f(x-y) dx \right) g(y) dy \\ &= \|f\|_1 \sum_{\mathbb{R}^n} g(y) dy = \|f\|_1 \|g\|_1. \end{aligned}$$

Therefore, by Tonelli's theorem, $F \in L^1(\mathbb{R}^n \times \mathbb{R}^n)$.

Then, by the same theorem,

$$\|F\|_1 = \sum_{\mathbb{R}^n} \left(\sum_{\mathbb{R}^n} F(x, y) dy \right) dx = \sum_{\mathbb{R}^n} \left(\sum_{\mathbb{R}^n} f(y) g(x-y) dy \right) dx$$

It follows that $\int_{\mathbb{R}} f(y)g(x-y)dy < \infty$ (3)

for a.e. x , the function

$$x \mapsto (f * g)(x) = \int_{\mathbb{R}^n} F(x, y) dy$$

is measurable and $\|f * g\|_1 = \|F\|_1 = \|f\|_1 \|g\|_1$.

Finally, for general $f, g \in L^1(\mathbb{R}^n)$,

$$\begin{aligned} \|f * g\|_1 &= \int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} f(y)g(x-y)dy \right| dx \\ &\leq \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} |f(y)| \cdot |g(x-y)| dy \right) dx \\ &= \| |f| * |g| \|_1 = \| |f| \|_1 \cdot \| |g| \|_1 = \|f\|_1 \|g\|_1. \end{aligned}$$

Next let us show that $f * g$ is well-defined for $f \in L^1(\mathbb{R}^n)$, $g \in L^p(\mathbb{R}^n)$, $1 \leq p \leq \infty$.

We need some preparation.

Lemma

Assume $\varphi: (a, b) \rightarrow \mathbb{R}$ is a convex function. Then φ is continuous and

$$\varphi(x) = \sup \{ \ell(x) : \varphi \geq \ell, \ell(s) = \alpha s + \beta \}$$

Proof

We claim that the function $t \mapsto \frac{\varphi(t) - \varphi(x)}{t - x}$, $t \neq x$, is increasing for all x . This is equivalent to the inequalities

$$\frac{\varphi(t) - \varphi(x)}{t - x} \leq \frac{\varphi(y) - \varphi(x)}{y - x} \leq \frac{\varphi(y) - \varphi(t)}{y - t}$$

for $a < x < t < y < b$. To prove these inequalities,

write $t = \lambda x + (1-\lambda)y$, $0 < \lambda < 1$. Then

$$\frac{\varphi(t) - \varphi(x)}{t - x} \leq \frac{\lambda \varphi(x) + (1-\lambda)\varphi(y) - \varphi(x)}{t - x} = \frac{\varphi(y) - \varphi(x)}{y - x},$$

$$\frac{\varphi(y) - \varphi(t)}{y - t} \geq \frac{\varphi(y) - \lambda \varphi(x) - (1-\lambda)\varphi(y)}{y - t} = \frac{\varphi(y) - \varphi(x)}{y - x},$$

proving the claim.

The claim implies that the left and right derivatives φ'_- and φ'_+ exist and

$$\frac{\varphi(t_1) - \varphi(x)}{t_1 - x} \leq \varphi'_-(x) \leq \varphi'_+(x) \leq \frac{\varphi(t_2) - \varphi(x)}{t_2 - x}$$

for $t_1 < x < t_2$. From the existence of φ'_- , φ'_+

we conclude that φ is continuous.

We see also that for $\ell(s) = \varphi'_+(x)(s-x) + \varphi(x)$

we have $\varphi \geq \ell$ and $\varphi(x) = \ell(x)$. $\square$

Jensen's inequality

Assume $(X, \mathcal{B}, \mu)$ is a probability measure space (so $\mu(X) = 1$), $\varphi: [0, +\infty) \rightarrow [0, +\infty)$ is a convex function. Then, for every integrable function $f: X \rightarrow [0, +\infty)$ we

$$\text{have } \varphi\left(\int_X f d\mu\right) \leq \int_X \varphi \circ f d\mu.$$

The same inequality holds for any measurable $f: X \rightarrow [0, +\infty]$ if

$\lim_{x \rightarrow +\infty} \varphi(x) = +\infty$ and we put $\varphi(+\infty) = +\infty$.

Proof

Note that as φ is continuous on $(0, +\infty)$, the function $\varphi \circ f$ is measurable.

If $\int_X f d\mu = 0$, so $f = 0$ a.e., then there is nothing to prove, as then

$$\varphi\left(\int_X f d\mu\right) = \varphi(0), \quad \int_X \varphi \circ f d\mu = \varphi(0) \mu(X) = \varphi(0).$$

Assume therefore that $f: X \rightarrow (0, +\infty)$

and $t = \int_X f d\mu$ satisfies $0 < t < \infty$.

Take a linear function $l(s) = \alpha s + \beta$

s.t. $\varphi \geq l$ on $(0, +\infty)$. Then also $\varphi(0) \geq l(0) = \beta$

(Indeed, for $s > 0$ we have

$$l(s) \leq \varphi(s) \leq (1-s)\varphi(0) + s\varphi(1),$$

and letting $s \downarrow 0$ we get $\varphi(0) \geq l(0)$.)

Then

$$\begin{aligned} l\left(\int_X f d\mu\right) &= \alpha \int_X f d\mu + \beta \\ &= \int_X l \circ f d\mu \leq \int_X \varphi \circ f d\mu. \end{aligned}$$

Taking the supremum over l as above, by the previous lemma we get

$$\varphi\left(\int_X f d\mu\right) \leq \int_X \varphi \circ f d\mu.$$

Assume now that $\lim_{x \rightarrow +\infty} \varphi(x) = +\infty$

and put $\varphi(+\infty) = +\infty$.

If $f: X \rightarrow [0, +\infty]$ and $\int_X f d\mu < \infty$
(so $f < +\infty$ a.e.), then the same proof
as above gives the required inequality.

It remains to show that if

$$\int_X f d\mu = +\infty, \text{ then also } \int_X \varphi \circ f d\mu = +\infty.$$

For this it suffices to show that there
is $\ell(s) = \alpha s + \beta$ s.t. $\ell \geq \varphi$ and $\alpha > 0$,
as then

$$\int_X \varphi \circ f d\mu \geq \alpha \int_X f d\mu + \beta = +\infty.$$

Such an ℓ exists by the proof of
the previous lemma: take $t > 0$
and find x s.t. $\varphi(x) > \varphi(t)$. Then

$$\varphi'_+(x) \geq \frac{\varphi(t) - \varphi(x)}{t - x} > 0 \quad \text{and}$$

we can take $\ell(s) = \varphi'_+(x)(s - x) + \varphi(x)$. □