

①

Lemma

Assume $f \in L^1(\mathbb{R}^n)$, $g \in L^p(\mathbb{R}^n)$ ($1 \leq p \leq \infty$).
 Then the function $y \mapsto f(y)g(x-y)$ is
 integrable for a.e. x , $f * g \in L^p(\mathbb{R}^n)$ and
 $\|f * g\|_p \leq \|f\|_1 \|g\|_p$.

Proof

For $p = \infty$ the lemma is easy, since for every $x \in \mathbb{R}^n$,
 $|f(y)g(x-y)| \leq |f(y)| \|g\|_\infty$ for a.e. y .

Assume $1 < p < \infty$ (for $p = 1$ we already know
 the lemma).

For everything except the last inequality
 we may assume that $f, g \geq 0$.

If $f = 0$ a.e., there is nothing to prove,
 so we may assume that $\|f\|_1 > 0$ and
 then replacing f by $\frac{f}{\|f\|_1}$, that $\|f\|_1 = 1$.

By Jensen's inequality,

$$\left(\int_{\mathbb{R}^n} f(y)g(x-y) dy \right)^p \leq \int_{\mathbb{R}^n} f(y)g(x-y)^p dy \\
= (f * g^p)(x).$$

As $g^p \in L^1(\mathbb{R}^n)$, it follows that

$$\int_{\mathbb{R}^n} f(y)g(x-y) dy < \infty \text{ for a.e. } x \text{ and} \\
\|f * g\|_p \leq \left(\int_{\mathbb{R}^n} (f * g^p)(x) dx \right)^{1/p} \\
= \left(\|f\|_1 \|g^p\|_1 \right)^{1/p} = \|g^p\|_1^{1/p} = \|g\|_p.$$

Finally, for general $f \in L^1(\mathbb{R}^n)$, $g \in L^p(\mathbb{R}^n)$
we have

$$|f * g| \leq |f| * |g|,$$

$$\text{so } \|f * g\|_p \leq \| |f| * |g| \|_p \leq \| |f| \|_1 \cdot \| |g| \|_p \\ = \|f\|_1 \cdot \|g\|_p. \quad \square$$

Note that ~~that~~

$$\int_{\mathbb{R}^n} f(y) g(x-y) dy = \int_{\mathbb{R}^n} f(x-y) g(y) dy,$$

$$\text{so } f * g = g * f.$$

Remark

More generally, for $\mu \in M(\mathbb{R}^n)$ and $g \in L^p(\mathbb{R}^n)$,
we can define $\mu * g = g * \mu \in L^p(\mathbb{R}^n)$ by

$$(\mu * g)(x) = \int_{\mathbb{R}^n} g(x-y) d\mu(y).$$

$$\text{Then } \|\mu * g\|_p \leq |\mu|(\mathbb{R}^n) \|g\|_p.$$

Proposition

$$\text{If } f, g \in L^1(\mathbb{R}^n), \text{ then } \widehat{f * g} = \widehat{f} \widehat{g}.$$

Proof

$$\widehat{f * g}(x) = \int_{\mathbb{R}^n} (f * g)(y) e^{-i\langle x, y \rangle} dy$$

$$= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} f(z) g(y-z) e^{-i\langle x, y \rangle} dz \right) dy$$

$$= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} f(z) e^{-i\langle x, z \rangle} g(y-z) e^{-i\langle x, y-z \rangle} dy \right) dz$$

$$= (2\pi)^n \widehat{f} \widehat{g}.$$

$\square$

What are convolutions good for?

(2)

Example

Consider $f = \frac{1}{\lambda_n(B_r(0))} \mathbb{1}_{B_r(0)}$. Then

$$\begin{aligned}(f * g)(x) &= \frac{1}{\lambda_n(B_r(0))} \int_{B_r(0)} g(x-y) dy \\ &= \frac{1}{\lambda_n(B_r(x))} \int_{B_r(x)} g(y) dy.\end{aligned}$$

For a multi-index $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}_+^n$,
write ∂^α for ~~$\frac{\partial^{\alpha_1 + \dots + \alpha_n}}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}}$~~ $\frac{\partial^{\alpha_1 + \dots + \alpha_n}}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}}$.

Denote by $L^1_{loc}(\mathbb{R}^n)$ the space of
Lebesgue measurable functions that are
integrable on every ball. We identify functions
that coincide a.e. (so, more pedantically,
 $L^1_{loc}(\mathbb{R}^n)$ is a space of equivalence classes
of functions). We have $L^p(\mathbb{R}^n) \subset L^1_{loc}(\mathbb{R}^n)$
for all $1 \leq p \leq \infty$.

Lemma
If $\phi \in C_c^\infty(\mathbb{R}^n)$ and $f \in L^1_{loc}(\mathbb{R}^n)$, then
 $\phi * f \in C^\infty(\mathbb{R}^n)$ and $\partial^\alpha(\phi * f) = (\partial^\alpha \phi) * f$.

Proof

It suffices to show that $\frac{\partial}{\partial x_i}(\phi * f) = \frac{\partial \phi}{\partial x_i} * f$.

Let $e_i = (0, \dots, 0, 1, 0, \dots, 0) \in \mathbb{R}^n$.

Then

$$\frac{\partial (\phi * f)}{\partial x_i}(x) = \frac{d}{dt} (\phi * f)(x + te_i)$$

$$= \frac{d}{dt} \int_{\mathbb{R}^n} \phi(x + te_i - y) f(y) dy$$

$$= \int_{\mathbb{R}^n} \frac{\partial \phi}{\partial x_i}(x - y) f(y) dy = \left(\frac{\partial \phi}{\partial x_i} * f \right)(x),$$

where we used the theorem on differentiation of integrals depending on parameters. $\Rightarrow$

By choosing suitable ϕ we can make $\phi * f$ close to f , as we will see shortly.

Def

A positive mollifier is a function $\phi \in C_c(\mathbb{R}^n)$

s.t. $\phi \geq 0$ and $\int_{\mathbb{R}^n} \phi(x) dx = 1$.

For a function ϕ on $\mathbb{R}^n$ and $\varepsilon > 0$, define

$$\phi^\varepsilon(x) = \varepsilon^{-n} \phi\left(\frac{x}{\varepsilon}\right).$$

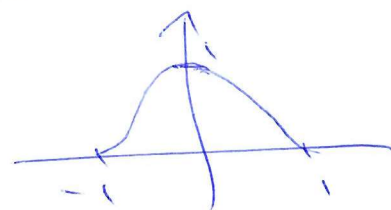
Note that if $\phi \in L^1(\mathbb{R}^n)$, $\int_{\mathbb{R}^n} \phi dx = 1$, then

$$\int_{\mathbb{R}^n} \phi^\varepsilon dx = 1.$$

Example

Consider the function h on $\mathbb{R}$ defined by

$$h(t) = \begin{cases} e^{-\frac{1}{1-t^2}}, & |t| < 1, \\ 0, & |t| \geq 1. \end{cases}$$



Then $h \in C_c^\infty(\mathbb{R})$. Hence $\phi(x) = c_n h(x)$ is a mollifier, where $c_n = \left(\int_{\mathbb{R}^n} h(|x|) dx \right)^{-1}$.

Proposition

Let $\phi \in L^1(\mathbb{R}^n)$ be s.t. $\phi \geq 0$ and $\int_{\mathbb{R}^n} \phi dx = 1$. ③

Then we have:

1) if $f \in C_0(\mathbb{R}^n)$, then $\phi^\varepsilon * f \in C_0(\mathbb{R}^n)$ and

$$\|\phi^\varepsilon * f - f\| \xrightarrow{\varepsilon \downarrow 0} 0 \quad (\text{uniform norm});$$

2) if $f \in L^p(\mathbb{R}^n)$, $1 \leq p < \infty$, then

$$\|\phi^\varepsilon * f - f\|_p \xrightarrow{\varepsilon \downarrow 0} 0.$$

Proof

Since $\|\phi^\varepsilon * f\| \leq \|f\|$, $\|\phi^\varepsilon * f\|_p \leq \|f\|_p$ and $C_c(\mathbb{R}^n)$ is dense in $C_0(\mathbb{R}^n)$ and $L^p(\mathbb{R}^n)$ ($1 \leq p < \infty$), it suffices to prove the proposition for $f \in C_c(\mathbb{R}^n)$. Let $\varepsilon > 0$ be s.t. $\text{supp } f \subset B_\varepsilon(0)$.

Assume first that ϕ is zero outside $B_R(0)$ for some $R > 0$. Then ϕ^ε is zero outside $B_{\varepsilon R}(0)$.

As f is uniformly continuous, for every $\delta > 0$ we can find $\varepsilon_0 > 0$ s.t. if $|x - y| < \varepsilon_0 R$, then

$$|f(x) - f(y)| < \delta.$$

Then, if $|x - y| < \varepsilon_0 R$, we have for ~~all~~ $\varepsilon > 0$ that

$$|(\phi^\varepsilon * f)(x) - (\phi^\varepsilon * f)(y)| = \left| \int_{\mathbb{R}^n} (f(x-z) - f(y-z)) \phi^\varepsilon(z) dz \right|$$

$$\leq \int_{\mathbb{R}^n} |f(x-z) - f(y-z)| \phi^\varepsilon(z) dz$$

$$\leq \delta \int_{\mathbb{R}^n} \phi^\varepsilon(z) dz = \delta.$$

This implies that $\phi^\varepsilon * f$ is continuous.

Similarly, for $x \in \mathbb{R}^n$ and $\varepsilon < \varepsilon_0$,

$$\begin{aligned}
|(\varphi^\varepsilon * f)(x) - f(x)| &= \left| \int_{\mathbb{R}^n} f(x-y) \varphi^\varepsilon(y) dy - f(x) \int_{\mathbb{R}^n} \varphi^\varepsilon(y) dy \right| \\
&= \left| \int_{\mathbb{R}^n} f(x-y) \varphi^\varepsilon(y) dy - f(x) \int_{\mathbb{R}^n} \varphi^\varepsilon(y) dy \right| \\
&\leq \int_{\mathbb{R}^n} |f(x-y) - f(x)| \varphi^\varepsilon(y) dy \\
&\leq \int_{\mathbb{R}^n} \varphi^\varepsilon(y) dy = 1, \text{ as } \varphi^\varepsilon \text{ is zero outside } B_{\varepsilon_0} \mathbb{R}
\end{aligned}$$

Therefore $\varphi^\varepsilon * f \xrightarrow{\varepsilon \downarrow 0} f$ uniformly on $\mathbb{R}^n$.

As $\text{supp}(\varphi^\varepsilon * f) \subset B_{\varepsilon R + r}(0)$, it follows also that $\varphi^\varepsilon * f \xrightarrow{\varepsilon \downarrow 0} f$ in $L^p(\mathbb{R}^n)$ for all $1 \leq p < \infty$.

For general φ , fix $\delta > 0$ and choose $R > 0$ s.t. $\int_{\mathbb{R}^n \setminus B_R(0)} \varphi dx < \delta$.

write $\varphi = \varphi_1 + c \varphi_2$, where

$$\varphi_1 = \varphi \mathbb{1}_{\mathbb{R}^n \setminus B_R(0)}, \quad c = \int_{B_R^*(0)} \varphi dx \quad (so \quad 1 - \delta < c < 1)$$

and $\varphi_2 = \frac{1}{c} \varphi \mathbb{1}_{B_R(0)}$. Then we know already that $\varphi_2^\varepsilon * f \rightarrow f$ in $C_0(\mathbb{R}^n)$

and $L^p(\mathbb{R}^n)$ ($1 \leq p < \infty$). Hence

$$\begin{aligned}
\|\varphi^\varepsilon * f - f\| &\leq \|\varphi_1^\varepsilon * f\| + c \|\varphi_2^\varepsilon * f - f\| + (1-c) \|f\| \\
&\leq \delta \|f\| + c \|\varphi_2^\varepsilon * f - f\| + \delta \|f\|.
\end{aligned}$$

As $\delta > 0$ was arbitrary, it follows that

$$\varphi^\varepsilon * f - f \xrightarrow{\varepsilon \downarrow 0} 0 \text{ in } C_0(\mathbb{R}^n). \text{ The}$$

same proof with $\|\cdot\|$ replaced by $\|\cdot\|_p$,
shows that $\varphi^\varepsilon * f - f \xrightarrow{\varepsilon \downarrow 0} 0$ in $L^p(\mathbb{R}^n)$,
($1 \leq p < \infty$). $\square$

Corollary

For any Radon measure μ on $\mathbb{R}^n$,
 $C_c^\infty(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n, d\mu)$ for $1 \leq p < \infty$.

Proof

We already know that $C_c(\mathbb{R}^n)$ is
dense in $L^p(\mathbb{R}^n, d\mu)$. On the other
hand, if $f \in C_c(\mathbb{R}^n)$, $\text{supp } f \subset B_r(0)$,
 $\varphi \in C_c^\infty(\mathbb{R}^n)$ is a positive mollifier,
 $\text{supp } \varphi \subset B_R(0)$, then $\varphi^\varepsilon * f \in C_c^\infty(\mathbb{R}^n)$,
 $\varphi^\varepsilon * f \rightarrow f$ uniformly on $\mathbb{R}^n$ and
 $\text{supp } (\varphi^\varepsilon * f) \subset B_{r+R}(0)$ if $\varepsilon \leq 1$,
hence $\varphi^\varepsilon * f \rightarrow f$ in $L^p(\mathbb{R}^n, d\mu)$. $\square$