

Corollary (Riemann-Lebesgue lemma)

①

If $f \in L^1(\mathbb{R}^n)$, then $\hat{f} \in C_0(\mathbb{R}^n)$.

Proof
We already know that $\hat{f} \in C_b(\mathbb{R}^n)$ (the space of bounded continuous functions) and $\|\hat{f}\|_\infty \leq \|f\|_1$.

Consider $\varphi(x) = \frac{1}{(2\pi)^{n/2}} e^{-\frac{|x|^2}{2}}$, so $\int_{\mathbb{R}^n} \varphi(x) dx = 1$.

We know that $\hat{\varphi} = \frac{1}{(2\pi)^{n/2}} \varphi \in C_0(\mathbb{R}^n)$.

Then $\hat{\varphi}^\varepsilon(x) = \hat{\varphi}(\varepsilon x) = \frac{1}{(2\pi)^{n/2}} \varphi(\varepsilon x)$, $\hat{\varphi}^\varepsilon \in C_0(\mathbb{R}^n)$.

As $\varphi^\varepsilon * f \xrightarrow{\varepsilon \downarrow 0} f$ in $L^1(\mathbb{R}^n)$, we have

$$\|\widehat{\varphi^\varepsilon * f} - \hat{f}\|_\infty \xrightarrow{\varepsilon \downarrow 0} 0.$$

But $\widehat{\varphi^\varepsilon * f} = (2\pi)^n \hat{\varphi}^\varepsilon \hat{f} \in C_0(\mathbb{R}^n)$,

hence $\hat{f} \in C_0(\mathbb{R}^n)$. $\square$

Remark

Another possibility is to approximate f by linear combinations of $\Pi_{[a,b]}$.

Note that for $\Pi_{[a,b]} \in L^1(\mathbb{R})$ we have

$$\widehat{\Pi_{a,b}}(x) = \frac{1}{2\pi} \int_a^b e^{-ixy} dy = \frac{e^{-iax} - e^{-ibx}}{2\pi i x} \xrightarrow{x \rightarrow \infty} 0.$$

Fourier inversion theorem

Assume $f \in L^1(\mathbb{R}^n)$ is s.t. $\hat{f} \in L^1(\mathbb{R}^n)$.

Then, for a.e. x , we have

$$f(x) = \int_{\mathbb{R}^n} \hat{f}(y) e^{i\langle x, y \rangle} dy.$$

Equivalently,

$$\hat{\hat{f}}(x) = \frac{1}{(2\pi)^n} f(-x).$$

Lemma

For any $f, g \in L^1(\mathbb{R}^n)$, we have

$$\int_{\mathbb{R}^n} \hat{f} g dx = \int_{\mathbb{R}^n} f \hat{g} dx.$$

(Note that $\hat{f} g \in L^1(\mathbb{R}^n)$, as $\hat{f}$ is bounded.)

Proof

This follows by Fubini's theorem:

$$\begin{aligned} \int_{\mathbb{R}^n} \hat{f}(x) g(x) dx &= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} f(y) g(x) e^{-i\langle x, y \rangle} dy \right) dx \\ &= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} f(y) g(x) e^{-i\langle x, y \rangle} dx \right) dy \\ &= \int_{\mathbb{R}^n} f(y) \hat{g}(y) dy. \quad \square \end{aligned}$$

Proof of the theorem

We already know that the theorem is true for $\phi(x) = \frac{1}{(2\pi)^{n/2}} e^{-\frac{|x|^2}{2}}$, hence also for $\phi \in \mathcal{S}$ (so $\phi^\varepsilon(x) = \varepsilon^{-n} \phi(x/\varepsilon)$).

Consider the function I_ε defined by ②

$$\begin{aligned} I_\varepsilon(x) &= \int_{\mathbb{R}^n} \widehat{\varphi^\varepsilon * f}(y) e^{i\langle x, y \rangle} dy \\ &= (2\pi)^n \int_{\mathbb{R}^n} \widehat{\varphi^\varepsilon}(y) \widehat{f}(y) e^{i\langle x, y \rangle} dy \end{aligned}$$

On the one hand,

$$(2\pi)^n \widehat{\varphi^\varepsilon}(y) = (2\pi)^n \widehat{\varphi}(\varepsilon y) = e^{-\frac{\varepsilon^2 |y|^2}{2}}.$$

As $\widehat{f} \in L^1(\mathbb{R}^n)$ by assumption, by the dominated convergence theorem we conclude that

$$I_\varepsilon(x) \rightarrow \int_{\mathbb{R}^n} \widehat{f}(y) e^{i\langle x, y \rangle} dy \quad \forall x \in \mathbb{R}^n.$$

On the other hand, applying the previous lemma to $g(y) = \widehat{\varphi^\varepsilon}(y) e^{i\langle x, y \rangle}$

(with x fixed) we have

$$\begin{aligned} I_\varepsilon(x) &= (2\pi)^n \int_{\mathbb{R}^n} g(y) \widehat{f}(y) dy \\ &= (2\pi)^n \int_{\mathbb{R}^n} \widehat{g}(y) f(y) dy \\ &= (2\pi)^n \int_{\mathbb{R}^n} \widehat{\widehat{\varphi^\varepsilon}(y) e^{i\langle x, y \rangle}}(y) f(y) dy \\ &= \int_{\mathbb{R}^n} \varphi^\varepsilon(y-x) f(y) dy \\ &= (\varphi^\varepsilon * f)(x). \end{aligned}$$

Therefore $I_\varepsilon = \varphi^\varepsilon * f \xrightarrow{\varepsilon \downarrow 0} f$ in $L^1(\mathbb{R}^n)$.

In particular $I_{y_n} \xrightarrow{n \rightarrow \infty} f$ in $L^1(\mathbb{R}^n)$,

hence $I_{y_{n_k}} \xrightarrow{n_k \rightarrow \infty} f$ ~~for some subsequence~~ a.e.
for some subsequence.

It follows that for a.e. x we have

$$f(x) = \int_{\mathbb{R}^n} \hat{f}(y) e^{i\langle x, y \rangle} dy$$

Corollary

If $f \in L^1(\mathbb{R}^n)$ and $\hat{f} = 0$, then $f = 0$ (a.e.)

Recall that a linear operator $U: H \rightarrow K$ between Hilbert spaces is called an isometry if

$$\|Ux\| = \|x\| \quad \forall x \in H.$$

Equivalently, by the polarization identity,
 $(Ux, Uy) = (x, y) \quad \forall x, y \in H.$

If U is in addition surjective, then it is called a unitary.

Theorem (Plancherel)

There is a unique unitary $U: L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n)$
s.t. $Uf = (2\pi)^{\frac{n}{2}} \hat{f}$ for all $f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$

Proof

Consider the subspace

$$V = \{ f \in L^2(\mathbb{R}^n) : f \in L^1(\mathbb{R}^n), \hat{f} \in L^2(\mathbb{R}^n) \}$$

of $L^2(\mathbb{R}^n)$.

It $f \in V$, then $\hat{f} \in L^2(\mathbb{R}^n)$, since ③

$$\|\hat{f}\|_2^2 \leq \|f\|_\infty \|f\|_1.$$

Moreover, by the lemma used in the proof of Fourier inversion, we have

$$\begin{aligned} \|\hat{f}\|_2^2 &= \int_{\mathbb{R}^n} \hat{f}(x) \overline{\hat{f}}(x) dx = \int_{\mathbb{R}^n} f(x) \overline{\hat{\hat{f}}}(x) dx \\ &= \int_{\mathbb{R}^n} f(x) \overline{\hat{f}}(-x) dx = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} f(x) \overline{f}(x) dx \\ &= \frac{1}{(2\pi)^n} \|f\|_2^2. \end{aligned}$$

This shows that we have an isometry

$$U: V \rightarrow L^2(\mathbb{R}^n), \quad Uf = (2\pi)^{n/2} \hat{f}.$$

The space V is dense in $L^2(\mathbb{R}^n)$, since it contains functions $\varphi^\varepsilon * f$ for $f \in C_c(\mathbb{R}^n)$ and $\varphi(x) = \frac{1}{(2\pi)^{n/2}} e^{-\frac{|x|^2}{2}}$, and these functions are dense in $L^2(\mathbb{R}^n)$, because $\varphi^\varepsilon * f \rightarrow f$ and $C_c(\mathbb{R}^n)$ is dense in $L^2(\mathbb{R}^n)$.

It follows that U extends to an isometric operator ~~U: L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n)~~ $U: L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n)$. see p. 4

Next, we claim that $Uf = (2\pi)^{n/2} \hat{f}$ for all $f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$. Indeed, for such f , we have $\varphi^\varepsilon * f \in V$ and $\varphi^\varepsilon * f \xrightarrow{\varepsilon \rightarrow 0} f$ in $L^2(\mathbb{R}^n)$. Hence, on the one hand,

$$U(\varphi^\varepsilon * f) \xrightarrow{\varepsilon \downarrow 0} Uf \text{ in } L^2(\mathbb{R}^n).$$

On the other hand, for all $x \in \mathbb{R}^n$,

$$\begin{aligned} (U(\varphi^\varepsilon * f))(x) &= (2\pi)^{\frac{3n}{2}} \widehat{\varphi^\varepsilon}(x) \widehat{f}(x) \\ &= (2\pi)^{\frac{n}{2}} e^{-\frac{\varepsilon^2 |x|^2}{2}} \widehat{f}(x). \end{aligned}$$

We see that $U(\varphi^\varepsilon * f) \xrightarrow{\varepsilon \downarrow 0} (2\pi)^{\frac{n}{2}} \widehat{f}$ pointwise and in $L^2(\mathbb{R}^n)$. Hence $Uf = (2\pi)^{\frac{n}{2}} \widehat{f}$.

It remains to check that U is surjective. If $f \in V$, then $Uf \in V \subset L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$, so

$$U^2 f = (2\pi)^{\frac{n}{2}} U \widehat{f} = (2\pi)^n \widehat{\widehat{f}},$$

hence $(U^2 f)(x) = f(-x)$ for a.e. x . Therefore

$$\text{Im } U \supset \text{Im } U^2 \supset V.$$

As U is isometric and $L^2(\mathbb{R}^n)$ is see p. 4 complete, $\text{Im } U$ is closed.

As V is dense in $L^2(\mathbb{R}^n)$, we conclude that $\text{Im } U = L^2(\mathbb{R}^n)$. □

For $f \in L^2(\mathbb{R}^n)$ we now define its Fourier transform as the function $\widehat{f} \in L^2(\mathbb{R}^n)$ given by

$$\widehat{f} = (2\pi)^{-\frac{n}{2}} Uf.$$

If $f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$, we therefore have

$$\widehat{f}(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} f(y) e^{-i\langle x, y \rangle} dy,$$

but if $f \in L^2(\mathbb{R}^n) \setminus L^1(\mathbb{R}^n)$, this integral does not make any sense.

In the proof we used the following two properties: ④

1) If H, K are Hilbert spaces, $V \subset H$ is a dense subspace and $U: V \rightarrow K$ is an isometry, then U extends uniquely to an isometry $U: H \rightarrow K$.

Namely, if $x \in H$, find a sequence $(x_n)_n$ in V converging to x . Then $(x_n)_n$ is Cauchy. Hence $(Ux_n)_n$ is also Cauchy, as

$$\|Ux_n - Ux_m\| = \|x_n - x_m\| \xrightarrow{n, m \rightarrow \infty} 0.$$

As K is complete, $(Ux_n)_n$ is convergent. We then define

$$Ux = \lim_{n \rightarrow \infty} Ux_n.$$

It is not difficult to check that U is well-defined, linear and isometric.

2) If H, K are Hilbert spaces and $U: H \rightarrow K$ is an isometry, then $\text{Im } U$ is closed in K .

Indeed, assume $y \in \overline{\text{Im } U}$, so there exist $x_n \in H$ s.t. $y = \lim_{n \rightarrow \infty} Ux_n$. Then $(Ux_n)_n$ is a Cauchy sequence, hence $(x_n)_n$ is a Cauchy sequence as well, as $\|x_n - x_m\| = \|Ux_n - Ux_m\|$.

Hence $x_n \rightarrow x \in H$ for some x .

Then $Ux = \lim_{n \rightarrow \infty} Ux_n = y$, so $y \in \text{Im } U$.