

Schwartz space

①

Proposition

Assume $f \in L^1(\mathbb{R}^n)$ and $x_j f \in L^1(\mathbb{R}^n)$ for some $1 \leq j \leq n$. Then

$$\widehat{\partial_j f} = -i \widehat{x_j f} \quad (\partial_j = \frac{\partial}{\partial x_j})$$

(here by $x_j f$ we mean the function $x \mapsto x_j f(x)$.)

Proof

Follows from the theorem on differentiation of parameter-dependent integrals. $\square$
we have a "dual" result:

Proposition

Assume $f \in C(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$ is such that $\partial_j f \in L^1(\mathbb{R}^n)$. Then

$$\widehat{\partial_j f} = i x_j \widehat{f}$$

Proof

Assume first that f is in addition compactly supported. Then by integration by parts we have

$$\int_{\mathbb{R}^n} \frac{\partial f}{\partial x_j}(y) e^{-i\langle x, y \rangle} dy = i x_j \int_{\mathbb{R}^n} f(y) e^{-i\langle x, y \rangle} dy$$

Hence

$$\int_{\mathbb{R}^n} \frac{\partial f}{\partial x_j}(y) e^{-i\langle x, y \rangle} dy = i x_j \int_{\mathbb{R}^n} f(y) e^{-i\langle x, y \rangle} dy,$$

so $\widehat{\partial_j f} = i x_j \widehat{f}$.

For general f , consider a positive mollifier $\phi \in C_c^\infty(\mathbb{R}^n)$. Define

$$\phi_k(x) = \phi\left(\frac{x}{k}\right).$$

Then $\phi_k f \in C_c(\mathbb{R}^n)$ and $\phi_k f \xrightarrow{x \rightarrow \infty} f$ pointwise and in $L^1(\mathbb{R}^n)$.

We also have

$$\partial_j(\phi_k f)(x) = \frac{1}{k} (\partial_j \phi)\left(\frac{x}{k}\right) f(x) + \phi_k(x) (\partial_j f)(x).$$

Since $\|(\partial_j \phi)\left(\frac{x}{k}\right)\| \leq \|\partial_j \phi\|_\infty$, we see that also $\partial_j(\phi_k f) \rightarrow \partial_j f$ pointwise and in $L^1(\mathbb{R}^n)$.

Hence $\widehat{\partial_j(\phi_k f)} \rightarrow \widehat{\partial_j f}$ uniformly on $\mathbb{R}^n$.

On the other hand,

$$\widehat{\partial_j(\phi_k f)} = i x_j \widehat{\phi_k f}$$

and $\widehat{\phi_k f} \rightarrow \widehat{f}$ uniformly on $\mathbb{R}^n$.

Hence $\widehat{\partial_j f} = i x_j \widehat{f}$. □

Corollary

If $f, \partial_j f \in L^1(\mathbb{R}^n)$, then

$$\widehat{\partial_j f} = i x_j \widehat{f} \quad \text{and} \quad x_j \widehat{f}(x) \xrightarrow{x \rightarrow \infty} 0.$$

~~Proof~~

Proof

Indeed, $x_j \widehat{f} = -i \widehat{\partial_j f} \in C_0(\mathbb{R}^n)$. □

we can apply these two propositions repeatedly and get the following:

Corollary

1) If $x^\alpha f \in L^1(\mathbb{R}^n)$ for all $|\alpha| \leq N$, then $\widehat{f} \in C^N(\mathbb{R}^n)$ and $\partial^\alpha \widehat{f} = (-i)^{|\alpha|} \widehat{x^\alpha f}$.

2) If $f \in C^N(\mathbb{R}^n)$ and $\widehat{\partial^\alpha f} \in L^1(\mathbb{R}^n)$ for all $|\alpha| \leq N$, then $\widehat{\partial^\alpha f} = i^{|\alpha|} x^\alpha \widehat{f}$

and hence $(1+|x|)^N \widehat{f}(x) \rightarrow 0$ as $|x| \rightarrow \infty$.

(Here $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}_+^n$, $|\alpha| = \alpha_1 + \dots + \alpha_n$,
 $x^\alpha = x_1^{\alpha_1} \dots x_n^{\alpha_n}$.)

Def

A function f on $\mathbb{R}^n$ is called a Schwartz function if $f \in C^\infty(\mathbb{R}^n)$ and $x^\alpha \partial^\beta f$ is bounded for all multi-indices α, β . The space $S(\mathbb{R}^n)$ of Schwartz functions is called a Schwartz space.

Note that for every $f \in C^\infty(\mathbb{R}^n)$ the following conditions are equivalent:

1) $x^\alpha \partial^\beta f$ is bounded for all α, β ;

2) $x^\alpha (\partial^\beta f)(x) \rightarrow 0$ as $|x| \rightarrow \infty$ for all α, β ;

3) $(1+|x|)^N \partial^\beta f$ is bounded for all $N \geq 1$ and all β .

Example

$C_c^\infty(\mathbb{R}^n) \subset S(\mathbb{R}^n)$, $e^{-a|x|^2} \in S(\mathbb{R}^n)$
for $a > 0$. If $f \in S(\mathbb{R}^n)$, then $x^\alpha f \in S(\mathbb{R}^n)$.

The product of two Schwartz functions is a Schwartz function.

clearly, $S(\mathbb{R}^n) \subset L^p(\mathbb{R}^n)$ for all $1 \leq p \leq \infty$.

From the Corollary above we conclude that if $f \in S(\mathbb{R}^n)$, then $\hat{f} \in S(\mathbb{R}^n)$.

By the Fourier inversion theorem we then get

Theorem

The Fourier transform maps $S(\mathbb{R}^n)$ onto $S(\mathbb{R}^n)$.

Remark

This gives another proof of the fact that the image of the Fourier transform

$L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n)$ is dense, which we needed to prove in Plancherel's theorem.

Remark

If $f \in C_c^\infty(\mathbb{R}^n)$, $f \neq 0$, then $\hat{f} \in S(\mathbb{R}^n)$, but $\hat{f}$ is never compactly supported, since it extends to an analytic function on $\mathbb{C}^n$: $\hat{f}(z) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} f(y) e^{-i\langle z, y \rangle} dy$.

Kolmogorov extension theorem

(9)

Assume X is a set and $(\mathcal{B}_n)_{n=1}^{\infty}$ is an increasing sequence of σ -algebras of subsets of X . Assume μ_n is a measure on $(X, \mathcal{B}_n)$ and

$$\mu_{n+1}|_{\mathcal{B}_n} = \mu_n \quad \forall n.$$

Can we define a measure μ on $(X, \mathcal{B})$, where $\mathcal{B} = \sigma(\bigcup_{n=1}^{\infty} \mathcal{B}_n)$ s.t. $\mu|_{\mathcal{B}_n} = \mu_n \quad \forall n$?

In general, no. But we have the following:

Theorem

In the above setting, assume in addition that $\mu_n(X) \geq 1 \quad \forall n$ and there is a collection of subsets $\mathcal{C} \subset \mathcal{B}$ s.t.:

$$(i) \quad \mu_n(A) = \sup \{ \mu_n(C) : C \subset A, C \in \mathcal{C} \cap \mathcal{B}_n \} \\ \forall A \in \mathcal{B}_n;$$

$$(ii) \quad \text{if } (C_n)_{n=1}^{\infty} \text{ is a sequence in } \mathcal{C} \text{ and } \bigcap_{n=1}^{\infty} C_n = \emptyset, \text{ then } \bigcap_{n=1}^N C_n = \emptyset \text{ for some } N.$$

Then there is a unique measure μ on $(X, \mathcal{B})$ s.t. $\mu|_{\mathcal{B}_n} = \mu_n$.

Proof

Consider $\mathcal{A} = \bigcup_{n=1}^{\infty} \mathcal{B}_n$.

This is an algebra of subsets of X , $\sigma(\mathcal{A}) = \mathcal{B}$, and we have a well-defined

function $\mu: \mathcal{A} \rightarrow [0, 1]$ s.t. $\mu(A) = \mu_n(A)$

if $A \in \mathcal{B}_n$.

we want to apply Carathéodory's extension theorem to extend μ from $\mathcal{A}$ to a measure on $\mathcal{B}$. For this we need to check that μ is a premeasure.

Assume $B = \bigcup_{n=1}^{\infty} B_n$, $B_i \cap B_m = \emptyset$ for $i \neq m$, $B, B_n \in \mathcal{A}$. We need to show that

$$\mu(B) = \sum_{n=1}^{\infty} \mu(B_n).$$

Define $A_n = B \setminus \bigcup_{k=1}^{n-1} B_k$. Then $A_n \in \mathcal{A}$,

$$\mu(B) = \sum_{k=1}^{n-1} \mu(B_k) + \mu(A_n).$$

Therefore we need to show that $\mu(A_n) \xrightarrow{n \rightarrow \infty} 0$.

Note that $A_n \downarrow \emptyset$ (that is, $A_1 \supset A_2 \supset \dots$),

$$\bigcap_{n=1}^{\infty} A_n = \emptyset.$$

Fix $\varepsilon > 0$. By assumption (i) we can

find $C_n \in \mathcal{C} \cap \mathcal{A}$ s.t.

$$\mu(A_n \setminus C_n) < \frac{\varepsilon}{2^n}.$$

$$C_n \subset A_n, \text{ as } \bigcap_{n=1}^{\infty} C_n = \emptyset.$$

Then $\bigcap_{n=1}^{\infty} C_n = \emptyset$, as $\bigcap_{n=1}^{\infty} A_n = \emptyset$.

By assumption (ii), we have $\bigcap_{n=1}^N C_n = \emptyset$

for some N . Then

$$A_N = \bigcup_{n=1}^N (A_N \setminus C_n) \subset \bigcup_{n=1}^N (A_n \setminus C_n).$$

$$\begin{aligned} \text{Hence } \mu(A_N) &\leq \sum_{n=1}^N \mu(A_n \setminus C_n) \\ &< \sum_{n=1}^N \frac{\varepsilon}{2^n} < \varepsilon. \end{aligned}$$

It follows that $\mu(A_n) < \varepsilon \quad \forall n \geq N$.

Hence $\mu(A_n) \xrightarrow{n \rightarrow \infty} 0$.

Thus, μ is a premeasure, hence (9) it extends to $\mathcal{B}$.

The extension is unique by (a corollary to) Dynkin's theorem. $\square$

Remark

If in addition $\mathcal{C}$ is closed under finite unions and countable intersections, then

$$\mu(A) = \sup \{ \mu(C) : C \subset A, C \in \mathcal{C} \}$$

$\forall A \in \mathcal{B}$. Indeed, consider the collection $\tilde{\mathcal{B}}$ of sets $A \in \mathcal{B}$ s.t.

$$\mu(A) = \sup \{ \mu(C) : C \subset A, C \in \mathcal{C} \},$$

$$\mu(A^c) = \sup \{ \mu(C) : C \subset A^c, C \in \mathcal{C} \}.$$

Then $\tilde{\mathcal{B}}$ is a σ -algebra by the same arguments as in the lecture on regularity of measures, and $A \in \tilde{\mathcal{B}}$. Hence $\tilde{\mathcal{B}} = \mathcal{B}$.