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Assume now we have a collection
 $(X_i, \mathcal{B}_i)_{i \in I}$ of measurable spaces
 (I can be infinite and uncountable)

Consider $X = \prod_{i \in I} X_i$.

Denote by $\prod_{i \in I} \mathcal{B}_i$ the σ -algebra generated
 by all sets of the form

$$\prod_{i \in F} A_i \times \prod_{j \in F^c} X_j,$$

where $F \subset I$ is finite, $A_i \in \mathcal{B}_i$ ($i \in F$).

Example

Consider a sequence $(X_n, d_n)_{n=1}^{\infty}$ of
 separable metric spaces.

Assume $d_n(x, y) \leq 1 \quad \forall x, y$
 (Any metric d can be replaced by such
 by defining $\tilde{d}(x, y) = \frac{d(x, y)}{1 + d(x, y)}$.)

Then $\prod_{n=1}^{\infty} X_n$ is a metric space with

$$d(x, y) = \sum_{n=1}^{\infty} \frac{1}{2^n} d_n(x_n, y_n),$$

where $x = (x_n)_{n=1}^{\infty} \in X$.

Given a sequence $(x(n))_{n=1}^{\infty}$ in X ,
 we have $x(n) \xrightarrow{n \rightarrow \infty} x$ if and only if

$$x(n)_k \xrightarrow{n \rightarrow \infty} x_k \quad \forall k.$$

Consider the Borel σ -algebras $B(X_n)$.

Then $\bigcap_{n=1}^{\infty} B(X_n) = B(X)$.

To see this, for every n , choose open sets $U_{n,k} \subset X_n$ ($k=1,2,\dots$) s.t. every open set in X_n is the union of some of $U_{n,k}$'s.

~~Lemma~~ This is possible by separability: take a dense countable subset of X_n and then all balls of rational radii with centers at points of this subset. Then every open subset of X is the union of sets of the form

$$U_{1,k_1} \times U_{2,k_2} \times \dots \times U_{n,k_n} \times \bigcap_{m=n+1}^{\infty} X_m.$$

Therefore such sets generate the σ -algebra $B(X)$, and as $U_{n,k}$ ($n \in \mathbb{N}$) generate $B(X_n)$, we conclude that $\bigcap_{n=1}^{\infty} B(X_n) = B(X)$.

In relation to this example, we will need the following:

Theorem (Tikhonov, also transcribed as Tychonoff)
Assume $((X_n, d_n))_{n=1}^{\infty}$ is a sequence of compact metric spaces. Then $\bigcap_{n=1}^{\infty} X_n$ (with metric as in the example) is compact.

Proof

If $(x(n))_{n=1}^{\infty}$ is a sequence in $\bigcap_{n=1}^{\infty} X_n$, then by compactness, for every k ,

there is a subsequence of $(x(n)_k)_{n=1}^\infty$ in X_k converging to some point. ②

By the diagonal argument we can therefore find a subsequence $(x(n_\ell))_{\ell=1}^\infty$ s.t. $x(n_\ell)_k \xrightarrow{\ell \rightarrow \infty} x_k$ for some $x_k \in X_k$, for every k . Then for $\underline{x} = (x_k)_{k=1}^\infty$ we have $x(n_\ell) \xrightarrow{\ell \rightarrow \infty} \underline{x}$. □

Return to a general collection $(X_i, \mathcal{B}_i)_{i \in I}$ of measurable spaces. Let us introduce the following notation:

For $F \subset G \subset I$, write

$$X_F = \prod_{i \in F} X_i, \quad X_G = \prod_{i \in G} X_i,$$

$\pi_{G,F} : X_G \rightarrow X_F$ for the projection map:

$$\pi_{G,F}((x_i)_{i \in G}) = (x_i)_{i \in F}.$$

Kolmogorov extension theorem

Assume $(X_i)_{i \in I}$ is a collection of metric spaces. Consider $X = \prod_{i \in I} X_i$, $\mathcal{B} = \prod_{i \in I} \mathcal{B}(X_i)$.

Assume for every finite $F \subset I$ we are given a regular Borel probability measure μ_F on X_F s.t.

$$(\pi_{G,F})_* \mu_G = \mu_F$$

for all finite $F \subset G \subset I$. Then there is

a unique probability measure μ on $(X, \mathcal{B})$ s.t.

$$(\pi_{I, F})_* \mu = \mu_F$$

for all finite $F \subset I$.

Proof

~~uniqueness of the probability measure~~
 It suffices to consider countable sets I , since every $A \in \prod_{i \in I} \mathcal{B}(X_i)$ has the form $\tilde{A} \times \prod_{i \in I \setminus J} \mathcal{B}(X_i)$ for a countable

subset $J \subset I$ and $\tilde{A} \in \prod_{i \in J} \mathcal{B}(X_i)$.

There is nothing to prove when I is finite.

Therefore assume $I = \mathbb{N}$.

We want to apply the theorem from the previous lecture to

$$\mathcal{B}_n = \left\{ A \times \prod_{k=n+1}^{\infty} X_k : A \in \prod_{k=1}^n \mathcal{B}(X_k) \right\}$$

and measures

$$\mu_n \left(A \times \prod_{k=n+1}^{\infty} X_k \right) = \mu_{x_1, \dots, x_n}(A).$$

As the collection $\mathcal{C}$ in that theorem we take all sets of the form

$$K \times X_{\mathbb{N} \setminus F}, \text{ where } K \subset X_F \text{ is compact.}$$

We need to show that if $(C_n)_{n=1}^{\infty}$ is a sequence in $\mathcal{C}$ s.t. $\bigcap_{n=1}^{\infty} C_n = \emptyset$, then $\bigcap_{i=1}^N C_n = \emptyset$ for some N .

Write

(3)

$$C_n = K_n \times X_{I \setminus F_n}, K_n \subset X_{F_n} \text{ compact.}$$

By discarding factors that correspond to indices not in $\bigcup_{n=1}^{\infty} F_n$, we may assume that

$$\bigcup_{n=1}^{\infty} F_n = I.$$

For every $i \in I$ choose n_i s.t. $i \in F_{n_i}$.

Consider the compact sets

$$R_i = \overline{\{F_{n_i}, i\}} (K_{n_i}) \subset X_i.$$

By Tychonoff's theorem, the set

$$R = \bigcap_{i \in I} R_i \subset X = X_I$$

is compact (in the metric defined as in the example before that theorem).

As $C_n \cap R$ are closed subsets of R and

$$\bigcap_{n=1}^{\infty} (C_n \cap R) = \emptyset, \text{ it follows that there}$$

is No s.t.

$$\bigcap_{n=1}^{N_0} (C_n \cap R) = \emptyset.$$

Define

$$N = \max \{ N_0, n_i : i \in \bigcup_{n=1}^{N_0} F_n \}.$$

We claim that $\bigcap_{n=1}^N C_n = \emptyset$. Assume this is not the case, so there exists

$$x \in \bigcap_{n=1}^{N_0} C_n \cap \bigcap_{i \in \bigcup_{n=1}^{N_0} F_n} C_{n_i}.$$

As $x \in C_{n_i}$ for $i \in \bigcup_{n=1}^{N_0} F_n$, we have $x_i \in \Omega_i = \overline{U_{i,1/2}}(C_{n_i})$.

Choose $x'_i \in \Omega_i$ for $i \notin \bigcup_{n=1}^{N_0} F_n$ and consider the point $x' \in \Omega$ defined by

$$x'_i = \begin{cases} x_i, & \text{if } i \in \bigcup_{n=1}^{N_0} F_n, \\ x'_i, & \text{otherwise.} \end{cases}$$

As $x \in \bigcap_{n=1}^{N_0} C_n$, we have $x' \in \bigcap_{n=1}^{N_0} C_n$,

so $x' \in \bigcap_{n=1}^{N_0} (C_n \cap \Omega)$, which is a contradiction. $\square$

Remark

If in addition the spaces X_i are separable, then we can also conclude that for every $A \in \mathcal{B} = \prod_{i \in I} \mathcal{B}(X_i)$ we have

$$\mu(A) = \sup \mu(C),$$

where the supremum is taken over all sets $C \subset A$ of the form

$$C = K \times \prod_{i \in I \setminus J} X_i,$$

where $|I \setminus J|$ is countable and $K \subset X_J$ is compact.