

Random variables and stochastic processes

①

Assume $(X, \mathcal{B}, P)$ is a probability measure space.

If $(Y, \mathcal{C})$ is a measurable space, a measurable map $f: X \rightarrow Y$ is called a random variable. For $A \in \mathcal{C}$, define

$$P(f \in A) \stackrel{\text{def}}{=} P(f^{-1}(A)) = (f_* P)(A),$$

the probability that f takes a value in A .

The measure $f_* P$ on $(Y, \mathcal{C})$ is called the probability distribution of f .

Def

A stochastic process is a collection $(f_t: X \rightarrow Y)_{t \in T}$ of random variables.

T stands for "time" and is typically $\mathbb{Z}, \mathbb{Z}_+, \mathbb{R}$, or $\mathbb{R}_+$.

Given different $t_1, \dots, t_n \in T$, we can consider the joint distribution of

$f_{t_1}, \dots, f_{t_n}$, the measure

$$\mu_{t_1, \dots, t_n} = (f_{t_1} \times \dots \times f_{t_n})_* P \text{ on } (Y^n, \mathcal{C}^n).$$

when is a collection of measures defined by a stochastic process?

Theorem

Assume T is a set and for all different elements $t_1, \dots, t_n \in T$ we are given a Borel probability measure $\mu_{t_1, \dots, t_n}$ on $\mathbb{R}^n$ s.t.

(i) if $\sigma \in S_n$ and $A_1, \dots, A_n \in \mathcal{B}(\mathbb{R}^n)$, then

$$\mu_{t_1, \dots, t_n}(A_1 \times \dots \times A_n) = \mu_{t_{\sigma(1)}, \dots, t_{\sigma(n)}}(A_{\sigma(1)} \times \dots \times A_{\sigma(n)}).$$

$$(ii) \mu_{t_1, \dots, t_n, s_1, \dots, s_m}(A_1 \times \dots \times A_n \times \underbrace{\mathbb{R} \times \dots \times \mathbb{R}}_m)$$

$$= \mu_{t_1, \dots, t_n}(A_1 \times \dots \times A_n).$$

Then there is a probability measure space

$(X, \mathcal{B}, \mathbb{P})$ and a stochastic process

$(f_t: X \rightarrow \mathbb{R})_{t \in T}$ s.t. $\mu_{t_1, \dots, t_n}$ is the

joint distribution of $f_{t_1}, \dots, f_{t_n}$.

Proof

For a finite set $F = \{t_1, \dots, t_n\} \subset T$, put

$\mu_F = \mu_{t_1, \dots, t_n}$. This is a Borel probability

measure on $\mathbb{R}^F = \prod_{t \in F} \mathbb{R}$, which by (i)

is independent of the ordering. Consider

$X \stackrel{\text{def}}{=} \mathbb{R}^T = \prod_{t \in T} \mathbb{R}$, $\mathcal{B} = \prod_{t \in T} \mathcal{B}(\mathbb{R})$. By (i)

and (ii), we can apply Kolmogorov's

consistency theorem to get a probability

measure $\mathbb{P}$ on X s.t. $(\pi_{T,F})_* \mathbb{P} = \mu_F$

for all finite FCT

(2)

Define $f_t: X \rightarrow \mathbb{R}$ by $f_t((x_s)_{s \in T}) \stackrel{\text{def}}{=} x_t$.

Fig

Remark

Instead of $\mathbb{R}$ we could have taken any complete separable metric space, as then the measures $\mu_t, \dots, \mu_n$ are regular. Or we could just require the measures $\mu_t, \dots, \mu_n$ to be regular.

Random variables $f_1, \dots, f_n: X \rightarrow \mathbb{R}$ are called mutually independent if

$$P(f_1 \in A_1, \dots, f_n \in A_n) = P(f_1 \in A_1) \dots P(f_n \in A_n).$$

$$\Leftrightarrow P\left(\bigcap_{k=1}^n f_k^{-1}(A_k)\right) = \prod_{k=1}^n P(f_k^{-1}(A_k))$$

for all $A_1, \dots, A_n \in \mathcal{B}(\mathbb{R})$. In other words, if μ_i is the probability distribution of f_i , then the joint distribution of $f_1, \dots, f_n$ is

$$\mu_1 \times \dots \times \mu_n.$$

For such measures the above theorem gives the following result: if we are given a Borel probability measure μ_t on $\mathbb{R}$ for every $t \in T$, then we get a unique measure $\mu = \prod_{t \in T} \mu_t$ on $\mathbb{R}^T$.

$$\text{s.t. } (\pi_T, F) * \mu = \prod_{t \in F} \mu_t \quad \forall \text{ finite FCT.}$$

Example

Consider the process of tossing a coin. Write 0 for tail and 1 for heads.

We can model the process as follows.

$$X = \prod_{n=0}^{\infty} \{0,1\}, \quad P = \prod_{n=0}^{\infty} \nu, \quad \text{where } \nu = \frac{1}{2}\delta_0 + \frac{1}{2}\delta_1, \\ (B = \prod_{n=0}^{\infty} \mathcal{P}(\{0,1\}))$$

$$f_n: X \rightarrow \{0,1\}, \quad f_n(x) = x_n$$

f_n is the result of n -th toss

While the Kolmogorov extension theorem requires some regularity, it turns out that infinite products of probability measures always exist:

Theorem

Assume $(X_i, B_i, \mu_i)_{i \in I}$ is a collection of probability measure spaces. Consider $X = \prod_{i \in I} X_i$, $B = \prod_{i \in I} B_i$. Then there exists a unique measure μ on (X, B) s.t.
 $(\pi_{I,F})_* \mu = \prod_{i \in F} \mu_i \quad \forall \text{ finite } F \subset I.$

Proof

Uniqueness follows, as usual, from Dynkin's theorem.

As in the proof of Kolmogorov's theorem, to prove existence it suffices to consider countable I .

Thus, we may assume that $I = \mathbb{N}$.

Consider the algebra $A \subset B$ of sets

of the form $A \times \prod_{k=n+1}^{\infty} X_k$, where $A \in \prod_{k=1}^n B_k$. (3)

Define $\mu \left(A \times \prod_{k=n+1}^{\infty} X_k \right) \stackrel{\text{def}}{=} \left(\prod_{k=1}^n \mu_k \right) (A)$.

As in the proof of the theorem that we used in Kolmogorov's extension theorem, all we need to do is to check that μ is a premeasure, and for this we need to check that if $(A_n)_{n=1}^{\infty}$ is a sequence in $\mathcal{A}$, $A_n \downarrow \emptyset$, then $\mu(A_n) \rightarrow 0$.

Assume this is not true, so that

$$\mu(A_n) \geq \varepsilon > 0 \quad \forall n$$

for some ε .

For every $n \geq 1$, denote by $X^{(n)}, \mathcal{A}^{(n)}, \mu^{(n)}$ the same objects as $X, \mathcal{A}, \mu$, but constructed for the sequence $(X_k, B_k, \mu_k)_{k=n+1}^{\infty}$.

For $x \in X$, and $A \in \mathcal{A}$, define

$$A^{(n)}(x) = \{ y \in X^{(n)} : (x, y) \in A \}$$

By the Fubini-Tonelli theorem, we have

$A^{(n)}(x) \in \mathcal{A}^{(n)}$ and

$$\mu(A) = \int_X \mu^{(n)}(A^{(n)}(x)) d\mu_1(x).$$

(The point is that as $A \in \mathcal{A}$, we effectively work only with finite products and so Fubini-Tonelli applies.)

We now apply this to the set A_n .

Define $F_n = \{x \in X_1 : \mu^{(1)}(A_n^{(1)}(x)) \geq \frac{\varepsilon}{2}\}$. Then

$$\begin{aligned}\varepsilon \leq \mu(A_n) &= \int_{X_1} \mu^{(1)}(A_n^{(1)}(x)) d\mu_1(x) \\ &= \int_{F_n} + \int_{X_1 \setminus F_n} \\ &\leq \mu_1(F_n) + \frac{\varepsilon}{2} \mu_1(X_1 \setminus F_n) \\ &\leq \mu_1(F_n) + \frac{\varepsilon}{2},\end{aligned}$$

so $\mu_1(F_n) \geq \frac{\varepsilon}{2}$.

As $F_1 \supset F_2 \supset \dots$, we conclude that $\mu_1(\bigcap_{n=1}^{\infty} F_n) \geq \frac{\varepsilon}{2}$. Take any $x_1 \in \bigcap_{n=1}^{\infty} F_n$.

We then have $A_n^{(1)}(x_1) \in A^{(1)}$,

$A_n^{(1)}(x_1) \neq \emptyset$ and $\mu^{(1)}(A_n^{(1)}(x_1)) \geq \frac{\varepsilon}{2} \forall n$.

We now apply the same procedure to this sequence, and so on.

By induction we thus construct

points $x_k \in X_k$ s.t.

$$\mu^{(k+1)}(\{y \in X^{(k+1)} : (x_1, \dots, x_k, y) \in A_n^{(1)}\}) \geq \frac{\varepsilon}{2^{k+1}}$$

for all k and n . Since every A_n has

the form $\tilde{A}_n \times \prod_{m=N_n+1}^{\infty} X_m$ for some N_n and

$\tilde{A}_n \in \prod_{m=1}^{N_n} B(X_m)$, it follows that

$$x = \text{def } (x_k)_{k=1}^{\infty} \in A_n \quad \forall n.$$

This contradicts the assumption $\bigcap_{n=1}^{\infty} A_n = \emptyset$. □