Compulsory assignment in MAT3400/4400, Spring 2024

The solutions must be submitted (via Canvas) by 14:30 on Thursday, March 14, 2024. Try to solve as many (sub)problems as possible, but a half is enough to pass.

Problem 1. Assume $(X, \mathcal{B}, \mu)$ is a measure space and $\mathcal{A} \subset \mathcal{B}$ is an algebra. Consider the collection $\mathcal{C} \subset \mathcal{B}$ of sets $C \in \mathcal{B}$ such that for every $\varepsilon > 0$ there is $A \in \mathcal{A}$ satisfying

$$\mu(A \Delta C) < \varepsilon,$$

where $A \Delta C = (A \setminus C) \cup (C \setminus A)$.

(a) Show that $\mathcal{C}$ is an algebra containing $\mathcal{A}$.

(b) Show that if $\mu(X) < \infty$, then $\mathcal{C}$ is a σ-algebra. Therefore if $\mathcal{A}$ generates $\mathcal{B}$ as a σ-algebra, then $\mathcal{C} = \mathcal{B}$.

Problem 2. Assume X is a set, $\mathcal{A}$ is an algebra of subsets of X and μ is a premeasure on $(X, \mathcal{A})$. Consider the outer measure μ^* on X defined by μ:

$$\mu^*(A) = \inf \left\{ \sum_{n=1}^{\infty} \mu(A_n) : A_n \in \mathcal{A},\ A \subset \bigcup_{n=1}^{\infty} A_n \right\}.$$

Recall that a subset $A \subset X$ is called Caratheodory measurable (with respect to μ^*) if

$$\mu^*(B) = \mu^*(B \cap A) + \mu^*(B \cap A^c) \quad \text{for all} \quad B \subset X.$$

By the Caratheodory theorem, the collection Σ of Caratheodory measurable sets forms a σ-algebra containing $\mathcal{A}$ and $\mu^*|_\Sigma$ is a measure on (X, Σ) that extends μ. We continue to denote the measure $\mu^*|_\Sigma$ by μ. Consider also the σ-algebra $\mathcal{B} \subset \Sigma$ generated by $\mathcal{A}$.

(a) Show that for every subset $A \subset X$ there is $B \in \mathcal{B}$ such that

$$A \subset B \quad \text{and} \quad \mu^*(A) = \mu(B).$$

(b) Assume $\mu(X) < \infty$ and consider the completion $(X, \bar{\mathcal{B}}, \bar{\mu})$ of $(X, \mathcal{B}, \mu|_{\mathcal{B}})$ (recall the lecture from 5.02), so $\bar{\mathcal{B}}$ is the σ-algebra generated by $\mathcal{B}$ and all subsets A of the sets $B \in \mathcal{B}$ such that $\mu(B) = 0$. Show that $\Sigma = \bar{\mathcal{B}}$ and $\mu = \bar{\mu}$ on Σ.

(c) Show that the same result as in (b) holds if we replace the condition $\mu(X) < \infty$ by the assumption that there exist $A_n \in \mathcal{A}$ such that $A_n \uparrow X$ and $\mu(A_n) < \infty$ for all n.

(This is essentially Problem 6.4 in the book.)

(In particular, if we start with the algebra $\mathcal{A}$ consisting of finite unions of the intervals $[a, b)$, $(-\infty, c)$ and $[d, +\infty)$ in $\mathbb{R}$ with the premeasure defined by $\lambda([a, b)) = b - a$, then the σ-algebra of Caratheodory measurable sets is exactly the σ-algebra of Lebesgue measurable sets.)

Problem 3. Assume X is a set and μ^* is a finite outer measure on X.

(a) Show that for any subsets $A, B, C \subset X$ we have

$$\mu^*(A \Delta C) \le \mu^*(A \Delta B) + \mu^*(B \Delta C).$$

Conclude that we can define an equivalence relation $\sim$ on the set $\mathcal{P}(X)$ of subsets of X by

$$A \sim B \quad \text{iff} \quad \mu^*(A \Delta B) = 0.$$

(b) Consider the quotient space $\mathcal{P} = \mathcal{P}(X)/\sim$ and let $\pi\colon \mathcal{P}(X) \to \mathcal{P}$ be the quotient map. Show that the following defines a metric on $\mathcal{P}$:

$$d(\pi(A), \pi(B)) = \mu^*(A \Delta B).$$

Show also that

$$|\mu^*(A) - \mu^*(B)| \leq d(\pi(A), \pi(B)).$$

Therefore we get a well-defined continuous map $\mathcal{P} \to [0, +\infty)$, $\pi(A) \mapsto \mu^*(A)$.

(c) Consider the σ-algebra Σ of Caratheodory measurable sets. Show that if $A \in \Sigma$ and $A \sim B$ for some $B \subset X$, then $B \in \Sigma$. (Equivalently, the measure space $(X, \Sigma, \mu^*|_\Sigma)$ is complete.)

(d) Prove that the metric space $(\mathcal{P}, d)$ is complete. Show also that if $\mathcal{B} \subset \mathcal{P}(X)$ is a σ-algebra, then $\pi(\mathcal{B})$ is closed in $\mathcal{P}$. Hint: if $(A_n)_n$ is a sequence such that

$$d(\pi(A_n), \pi(A_{n+1})) < \frac{1}{2^n},$$

then $\lim_n \pi(A_n) = \lim_n \pi(B_n) = \pi(A)$, where

$$B_n = \bigcup_{m=n}^{\infty} A_m \quad \text{and} \quad A = \bigcap_{n=1}^{\infty} B_n = \bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} A_m.$$

(e) Assume now that $\mathcal{A}$ is an algebra of subsets of X, μ is a finite premeasure on $(X, \mathcal{A})$ and μ^* is the outer measure on X defined by μ. Conclude from the above results (or prove from scratch) that a subset $B \subset X$ is Caratheodory measurable if and only if $\pi(B) \in \overline{\pi(\mathcal{A})}$, that is, if and only if for every $\varepsilon > 0$ there is $A \in \mathcal{A}$ such that $\mu^*(A \Delta B) < \varepsilon$.

(This gives another interpretation of the Caratheodory extension theorem for finite premeasures: it is basically the extension by continuity of μ to a completion of $\mathcal{A}/\sim$.)