Solution to the compulsory assignment in MAT3400/4400, Spring 2024

Problem 1. Assume $(X, \mathcal{B}, \mu)$ is a measure space and $\mathcal{A} \subset \mathcal{B}$ is an algebra. Consider the collection $\mathcal{C} \subset \mathcal{B}$ of sets $C \in \mathcal{B}$ such that for every $\varepsilon > 0$ there is $A \in \mathcal{A}$ satisfying

$$\mu(A\Delta C) < \varepsilon,$$

where $A\Delta C = (A \setminus C) \cup (C \setminus A)$.

(a) Show that $\mathcal{C}$ is an algebra containing $\mathcal{A}$.

Solution. It is clear by definition that $\mathcal{A} \subset \mathcal{C}$. In particular, $\emptyset \in \mathcal{C}$. If $C \in \mathcal{C}$, $\varepsilon > 0$ and $A \in \mathcal{A}$ satisfies $\mu(A\Delta C) < \varepsilon$, then $\mu(A^c\Delta C^c) < \varepsilon$, since $A\Delta C = A^c\Delta C^c$. Hence $C^c \in \mathcal{C}$.

It remains to show that $\mathcal{C}$ is closed under finite unions. Take $C_1, C_2 \in \mathcal{C}$, fix $\varepsilon > 0$ and choose $A_1, A_2 \in \mathcal{A}$ such that $\mu(A_1\Delta C_1) < \varepsilon$ and $\mu(A_2\Delta C_2) < \varepsilon$. Since

$$(A_1 \cup A_2)\Delta(C_1 \cup C_2) \subset (A_1\Delta C_1) \cup (A_2\Delta C_2),$$

we have

$$\mu((A_1 \cup A_2)\Delta(C_1 \cup C_2)) \leq \mu(A_1\Delta C_1) + \mu(A_2\Delta C_2) < 2\varepsilon.$$

It follows that $C_1 \cup C_2 \in \mathcal{C}$. □

(b) Show that if $\mu(X) < \infty$, then $\mathcal{C}$ is a σ-algebra. Therefore if $\mathcal{A}$ generates $\mathcal{B}$ as a σ-algebra, then $\mathcal{C} = \mathcal{B}$.

Solution. We need to show that $\mathcal{C}$ is closed under countable unions. Take a sequence $(C_n)_{n=1}^\infty$ in $\mathcal{C}$ and put $C = \cup_{n=1}^\infty C_n$. Consider $B_n = \cup_{k=1}^n C_k$. By (a) we know that $B_n \in \mathcal{C}$. Fix $\varepsilon > 0$. As $B_n \uparrow C$, we have $\mu(B_n) \nearrow \mu(C)$. Hence we can find n such that $\mu(B_n) > \mu(C) - \varepsilon$, equivalently, $\mu(C \setminus B_n) < \varepsilon$. Choose $A \in \mathcal{A}$ such that $\mu(A\Delta B_n) < \varepsilon$. We have

$$A\Delta C \subset (A\Delta B_n) \cup (C \setminus B_n).$$

Hence

$$\mu(A\Delta C) \leq \mu(A\Delta B_n) + \mu(C \setminus B_n) < 2\varepsilon.$$

It follows that $C \in \mathcal{C}$. □

Problem 2. Assume X is a set, $\mathcal{A}$ is an algebra of subsets of X and μ is a premeasure on $(X, \mathcal{A})$. Consider the outer measure μ^* on X defined by μ:

$$\mu^*(A) = \inf\left\{\sum_{n=1}^\infty \mu(A_n) : A_n \in \mathcal{A},\ A \subset \bigcup_{n=1}^\infty A_n\right\}.$$

Recall that a subset $A \subset X$ is called Caratheodory measurable (with respect to μ^*) if

$$\mu^*(B) = \mu^*(B \cap A) + \mu^*(B \cap A^c) \quad \text{for all} \quad B \subset X.$$

By the Caratheodory theorem, the collection Σ of Caratheodory measurable sets forms a σ-algebra containing $\mathcal{A}$ and $\mu^*|_\Sigma$ is a measure on (X, Σ) that extends μ. We continue to denote the measure $\mu^*|_\Sigma$ by μ. Consider also the σ-algebra $\mathcal{B} \subset \Sigma$ generated by $\mathcal{A}$.

(a) Show that for every subset $A \subset X$ there is $B \in \mathcal{B}$ such that

$$A \subset B \quad \text{and} \quad \mu^*(A) = \mu(B).$$

Solution. If $\mu^*(A) = +\infty$, we can take $B = X$. Assume now that $\mu^*(A) < \infty$. For every n, we can find sets $A_{nk} \in \mathcal{A}$ ($k \geq 1$) such that

$$A \subset \bigcup_{k=1}^{\infty} A_{nk} \quad \text{and} \quad \mu^*(A) + \frac{1}{n} > \sum_{k=1}^{\infty} \mu(A_{nk}).$$

Consider the sets $B_n = \cup_{k=1}^{\infty} A_{nk} \in \mathcal{B}$. Then

$$A \subset B_n \quad \text{and} \quad \mu^*(A) + \frac{1}{n} > \sum_{k=1}^{\infty} \mu(A_{nk}) \geq \mu(B_n).$$

It follows that for $B = \cap_{n=1}^{\infty} B_n$ we have

$$A \subset B \quad \text{and} \quad \mu^*(A) + \frac{1}{n} > \mu(B) \quad \text{for all} \quad n.$$

Hence $\mu^*(A) \geq \mu(B)$. On the other hand, $\mu^*(A) \leq \mu^*(B) = \mu(B)$. Thus, $\mu^*(A) = \mu(B)$. □

(b) Assume $\mu(X) < \infty$ and consider the completion $(X, \bar{\mathcal{B}}, \bar{\mu})$ of $(X, \mathcal{B}, \mu|_{\mathcal{B}})$ (recall the lecture from 5.02), so $\bar{\mathcal{B}}$ is the σ-algebra generated by $\mathcal{B}$ and all subsets A of the sets $B \in \mathcal{B}$ such that $\mu(B) = 0$. Show that $\Sigma = \bar{\mathcal{B}}$ and $\mu = \bar{\mu}$ on Σ.

Solution. Take $A \in \Sigma$. By part (a) we can find $B \in \mathcal{B}$ such that $A \subset B$ and $\mu(A) = \mu^*(A) = \mu(B)$. Then $\mu(B \setminus A) = 0$. Applying (a) to $B \setminus A$, we can find $C \in \mathcal{B}$ such that $B \setminus A \subset C$ and $\mu(C) = 0$. This shows that $A = B \setminus (B \setminus A) \in \bar{\mathcal{B}}$ and $\bar{\mu}(A) = \mu(B) = \mu(A)$.

We have shown that $\Sigma \subset \bar{\mathcal{B}}$. For the opposite inclusion it suffices to check that if $B \in \mathcal{B}$, $\mu(B) = 0$ and $A \subset B$, then $A \in \Sigma$. A formally stronger statement is that if $A \subset X$ and $\mu^*(A) = 0$, then $A \in \Sigma$. In order to prove this, take any subset $C \subset X$. Then $\mu^*(C \cap A) \leq \mu^*(A) = 0$. Therefore

$$\mu^*(C \cap A) + \mu^*(C \cap A^c) = \mu^*(C \cap A^c) \leq \mu^*(C).$$

Since the opposite inequality always holds, we conclude that $A \in \Sigma$. □

(c) Show that the same result as in (b) holds if we replace the condition $\mu(X) < \infty$ by the assumption that there exist $A_n \in \mathcal{A}$ such that $A_n \uparrow X$ and $\mu(A_n) < \infty$ for all n.

Solution. The first part of the proof of (b) needed only that $\mu(A) < \infty$ rather than that $\mu(X) < \infty$. Hence, by that proof we can conclude that for every $A \in \Sigma$ and all $n \geq 1$ we have $A \cap A_n \in \bar{\mathcal{B}}$ and $\bar{\mu}(A \cap A_n) = \mu(A \cap A_n)$. As $(A \cap A_n) \uparrow A$, it follows that $A \in \bar{\mathcal{B}}$ and $\bar{\mu}(A) = \mu(A)$.

The second part of the proof of (b) didn't need any finiteness at all. □

Problem 3. Assume X is a set and μ^* is a finite outer measure on X.

(a) Show that for any subsets $A, B, C \subset X$ we have

$$\mu^*(A \Delta C) \leq \mu^*(A \Delta B) + \mu^*(B \Delta C).$$

Conclude that we can define an equivalence relation $\sim$ on the set $\mathcal{P}(X)$ of subsets of X by

$$A \sim B \quad \text{iff} \quad \mu^*(A \Delta B) = 0.$$

Solution. The inequality in the formulation is immediate from the inclusion $A\Delta C \subset (A\Delta B) \cup (B\Delta C)$. Next, it is obvious that for all subsets $A, B \subset X$ we have $A \sim A$ and if $A \sim B$, then $B \sim A$. Therefore we only need to check transitivity. Assume $A \sim B$ and $B \sim C$. Then

$$\mu^*(A\Delta C) \le \mu^*(A\Delta B) + \mu^*(B\Delta C) = 0,$$

hence $A \sim C$. □

(b) Consider the quotient space $\mathcal{P} = \mathcal{P}(X)/\sim$ and let $\pi\colon \mathcal{P}(X) \to \mathcal{P}$ be the quotient map. Show that the following defines a metric on $\mathcal{P}$:

$$d(\pi(A), \pi(B)) = \mu^*(A\Delta B).$$

Show also that

$$|\mu^*(A) - \mu^*(B)| \le d(\pi(A), \pi(B)).$$

Conclude that we get a well-defined continuous map $\mathcal{P} \to [0, +\infty)$, $\pi(A) \mapsto \mu^*(A)$.

Solution. The function d on $\mathcal{P} \times \mathcal{P}$ is well-defined, since if $A \sim A'$ and $B \sim B'$, then

$$\mu^*(A'\Delta B') \le \mu(A'\Delta A) + \mu^*(A\Delta B) + \mu^*(B\Delta B') = \mu^*(A\Delta B)$$

and for the same reasons $\mu^*(A\Delta B) \le \mu^*(A'\Delta B')$, hence $\mu^*(A\Delta B) = \mu^*(A'\Delta B')$.

The triangle inequality for d follows from (a). The remaining conditions on a metric – symmetry ($d(x, y) = d(y, x)$) and positivity ($d(x, y) \ge 0$ and $d(x, y) = 0$ if and only if $x = y$) – are immediate by definition. Therefore $(\mathcal{P}, d)$ is a metric space.

Next, for any $A, B \subset X$ we have $A \subset B \cup (A\Delta B)$. Hence

$$\mu^*(A) \le \mu^*(B) + \mu^*(A\Delta B).$$

For the same reason $\mu^*(B) \le \mu^*(A) + \mu^*(A\Delta B)$. Therefore

$$|\mu^*(A) - \mu^*(B)| \le \mu^*(A\Delta B),$$

proving the inequality in the formulation. This shows in particular that $\mu^*(A) = \mu^*(B)$ if $A \sim B$. It follows that $\pi(A) \mapsto \mu^*(A)$ is a well-defined function on $\mathcal{P}$.

□

(c) Consider the σ-algebra Σ of Caratheodory measurable sets. Show that if $A \in \Sigma$ and $A \sim B$ for some $B \subset X$, then $B \in \Sigma$. (Equivalently, the measure space $(X, \Sigma, \mu^*|_\Sigma)$ is complete.)

Solution. This can be checked using the definition of Caratheodory measurability, but we can also refer to the second part of the solution to Problem 2(b): as both $A \setminus B$ and $B \setminus A$ have outer measure zero, that part proves that both $A \setminus B$ and $B \setminus A$ lie in Σ, hence $B = (A \setminus (A \setminus B)) \cup (B \setminus A)$ lies in Σ as well. □

(d) Prove that the metric space $(\mathcal{P}, d)$ is complete. Show also that if $\mathcal{B} \subset \mathcal{P}(X)$ is a σ-algebra, then $\pi(\mathcal{B})$ is closed in $\mathcal{P}$. Hint: if $(A_n)_n$ is a sequence such that

$$d(\pi(A_n), \pi(A_{n+1})) < \frac{1}{2^n},$$

then $\lim_n \pi(A_n) = \lim_n \pi(B_n) = \pi(A)$, where

$$B_n = \bigcup_{m=n}^{\infty} A_m \quad \text{and} \quad A = \bigcap_{n=1}^{\infty} B_n = \bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} A_m.$$

Solution. Assume $(A_n)_{n=1}^\infty$ is a sequence in $\mathcal{P}(X)$ such that $(\pi(A_n))_n$ is a Cauchy sequence in $\mathcal{P}$. By passing to a subsequence we may assume that $d(\pi(A_n), \pi(A_{n+1})) < \frac{1}{2^n}$. Consider $B_n = \bigcup_{m=n}^\infty A_m$. Then $A_n \subset B_n$ and $B_n \setminus A_n \subset \cup_{m=n}^\infty (A_{m+1} \setminus A_m)$. Hence

$$d(\pi(A_n), \pi(B_n)) \le \mu^*\left(\bigcup_{m=n}^\infty (A_{m+1} \setminus A_m)\right) \le \sum_{m=n}^\infty \mu^*(A_{m+1} \setminus A_m) < \sum_{m=n}^\infty \frac{1}{2^m} = \frac{1}{2^{n-1}}.$$

Therefore if the sequence $(\pi(B_n))_n$ converges, then $(\pi(A_n))_n$ converges to the same point.

Consider $B = \cap_{n=1}^\infty B_n$, so that $B_n \downarrow B$. Then, for every n, we have $B \subset B_n$ and

$$B_n \setminus B = \bigcup_{m=n}^\infty (B_m \setminus B_{m+1}) \subset \bigcup_{m=n}^\infty (A_m \setminus A_{m+1}).$$

By the same computation as above we then get $d(\pi(B_n), \pi(B)) < \frac{1}{2^{n-1}}$, so $\pi(B_n) \to \pi(B)$.

This proves completeness of $(\mathcal{P}, d)$. The proof shows that every Cauchy sequence $(\pi(A_n))_n$ converges to a point $\pi(A)$ such that A lies in the σ-algebra generated by the sets A_n. Hence $\pi(\mathcal{B})$ is closed for every σ-algebra $\mathcal{B}$. □

(e) Assume now that $\mathcal{A}$ is an algebra of subsets of X, μ is a finite premeasure on $(X, \mathcal{A})$ and μ^* is the outer measure on X defined by μ. Conclude from the above results (or prove from scratch) that a subset $B \subset X$ is Caratheodory measurable if and only if $\pi(B) \in \overline{\pi(\mathcal{A})}$, that is, if and only if for every $\varepsilon > 0$ there is $A \in \mathcal{A}$ such that $\mu^*(A \Delta B) < \varepsilon$.

Solution. By Problem 2(b) and the definition of $\mathcal{P}$ we have $\pi(\Sigma) = \pi(\bar{\mathcal{B}}) = \pi(\mathcal{B})$. By part (d), the set $\pi(\mathcal{B})$ is closed, and by Problem 1(b) the set $\pi(\mathcal{A})$ is dense in $\pi(\mathcal{B})$. Hence $\pi(\Sigma) = \overline{\pi(\mathcal{A})}$. By part (c) it follows that $B \in \Sigma$ if and only if $\pi(B) \in \overline{\pi(\mathcal{A})}$. □