

MAT 4410 : Advanced linear analysis

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B. Priver "Analysis tools with examples"

G. Pedersen "Analysis now"

D. McDonald, U. Weiss "A course in real analysis"

A. Kolmogorov, S. Fomin "Elements of the theory of functions and functional analysis"

18.08.14 Lecture 1

Three fundamental principles of functional analysis

I. Hahn-Banach theorem.

Let X be a real vector space.

A Minkowski functional is a function

$m: X \rightarrow \mathbb{R}$ s.t. $m(x+y) \leq m(x) + m(y)$

and $m(tx) = t m(x) \quad \forall t \geq 0$.

Theorem (Hahn-Banach)

Let X be a real vector space, $Y \subset X$ a subspace, m a Minkowski functional on X , $f: Y \rightarrow \mathbb{R}$

a linear functional s.t. $f(y) \leq m(y) \quad \forall y \in Y$.

Then f can be extended to a linear functional F on X s.t. $F(x) \leq m(x) \quad \forall x \in X$.

Proof

Assume first that $X = Y \oplus \mathbb{R}x_0$.

For some $x_0 \in X$. To extend f we need

to specify $c = F(x_0)$. The restriction on c is that

$$(*) \quad f(y) + \lambda c \leq m(y + \lambda x_0) \quad \forall y \in Y \quad \forall \lambda \in \mathbb{R}.$$

For $\lambda = 0$ this is true for any c . If $\lambda \neq 0$, then

(*) gives

$$c \leq \frac{m(y + \lambda x_0) - f(y)}{\lambda} = m\left(\frac{y}{\lambda} + x_0\right) - f\left(\frac{y}{\lambda}\right),$$

and if $\lambda < 0$, then (*) gives

$$c \geq -m\left(-\frac{y}{\lambda} + x_0\right) - f\left(-\frac{y}{\lambda}\right).$$

Therefore to show that c satisfying (*) exists it suffices to show that

$$\inf_{y \in Y} [m(y + x_0) - f(y)] \geq \sup_{y \in Y} (-m(y - x_0) - f(y)) \\ = \sup_{z \in Y} (-m(z - x_0) + f(z)),$$

as equivalently, that

$$-m(z - x_0) + f(z) \leq m(y + x_0) - f(y) \quad \forall y, z \in Y.$$

But this is true, as

$$f(z) + f(y) \leq f(y + z) \leq m(y + z) \\ \leq m(y + x_0) + m(z - x_0).$$

The general case follows from the above by Fenchel's lemma. $\square$

There are several (more or less) equivalent versions of the Hahn-Banach theorem. Seminorms.

Theorem
Let X be a complex or real vector space,
 p a seminorm on X , $Y \subset X$ a subspace,
 f a linear functional on Y s.t. $|f(y)| \leq p(y) \quad \forall y \in Y$.

Then there exists an extension of f to a
linear functional F on X s.t. $|F(x)| \leq p(x) \forall x \in X$. ②

Proof

Consider the complex case (the real case
follows easily from the previous theorem)
let us view X as a real vector space,
which we denote by $X_{\mathbb{R}}$.

Consider the linear functional

$$g: X_{\mathbb{R}} \rightarrow \mathbb{R}, \quad g(y) = \operatorname{Re} f(y).$$

Then $g(y) \leq |f(y)| \leq p(y) \forall y \in Y$.

hence g extends to a linear functional

$$G: X_{\mathbb{R}} \rightarrow \mathbb{R} \quad \text{s.t.} \quad G(x) \leq p(x) \quad \forall x \in X.$$

Define $F: X \rightarrow \mathbb{C}$ by

$$F(x) = G(x) - i G(ix).$$

Then F is a real linear functional on $X_{\mathbb{R}}$,

$$\text{and } F(ix) = G(ix) + i G(x) = i(G(x) - i G(ix)) = i F(x).$$

hence F is a complex linear functional on X .

Next, F extends f , since if

$$f(y) = a + ib, \text{ then } f(iy) = ia - b, \text{ so}$$

$$\begin{aligned} f(y) &= \operatorname{Re} f(y) - i \operatorname{Re} f(iy) = g(y) - i g(iy) \\ &= G(y) - i G(iy) = F(y). \end{aligned}$$

(In other words, a complex linear functional
is completely determined by its real part)

It remains to check that $|F(x)| \leq p(x)$.