

③ Fix $x \in X$. Choose $z \in \mathbb{T} = \{w \in \mathbb{C} \mid |w| = 1\}$ s.t. $z F(x) > 0$. Then

$$|F(x)| = z F(x) = F(zx) \in p(zx) = p(x).$$

Corollary

Let X be a complex or real normed vector space, $x_0 \in X \setminus \{0\}$. Then there exists $f \in X^*$ s.t. $\|f\| = 1$ and $f(x_0) = \|x_0\|$.

Proof

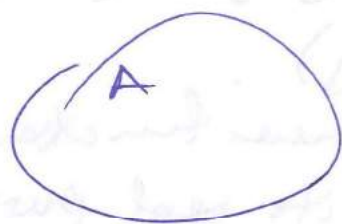
Apply the previous theorem to $Y = Kx_0$ ($K = \mathbb{C}$ or $K = \mathbb{R}$), $p(x) = \|x\|$, and the linear functional $Y \rightarrow K, \lambda x_0 \mapsto \lambda$.

Theorem (the Hahn-Banach ~~separation~~ separation theorem)

Let X be a real normed space, $A, B \subset X$ be ^{nonempty} convex sets s.t. $A \cap B = \emptyset$ and at least one of the sets has nonempty interior. Then there exists a nonzero

$f \in X^*$ and $c \in \mathbb{R}$ s.t.
 $f(a) \leq c \leq f(b) \quad \forall a \in A, b \in B$.

$$x_0 \in \text{Ker } f = \{x \mid f(x) = c\}$$



$$x_0 \in \text{Ker } f = \{x \mid f(x) = c\}$$

Remark / exercise

①

Same $f \neq 0$, $f(\text{int } A)$ and $f(\text{int } B)$ are open, where

$$f(a) < c < f(b) \quad \forall a \in \text{int } A \quad \forall b \in \text{int } B.$$

Proof

Consider first the case when $0 \in \text{int } A$ and $b = \{b\}$. Define

$$p_A(x) = \inf \{t > 0 \mid \frac{x}{t} \in A\}$$

Exercise

Show that p_A is a Minkowski functional,

Show also that ~~$p_A: X \rightarrow \mathbb{R}$~~

$$\{x \mid p_A(x) < 1\} \subset A \subset \{x \mid p_A(x) \leq 1\}.$$

Put $Y = \mathbb{R}b$ and define $g: Y \rightarrow \mathbb{R}$ by $g(b) = 1$.

Then $g(\lambda b) = \lambda g(b) = \lambda \leq \lambda p_A(b) = p_A(\lambda b)$ for $\lambda \geq 0$

and $g(\lambda b) = \lambda \leq 0 \leq p_A(\lambda b)$ for $\lambda \leq 0$.

hence we can extend g to a linear functional $f: X \rightarrow \mathbb{R}$ s.t. $f(x) \leq p_A(x) \quad \forall x \in X$.

Then $f(a) \leq p_A(a) \leq 1 = f(b)$ for any $a \in A$.

$$f(a) \leq p_A(a) \leq 1 = f(b) \quad \text{for any } a \in A.$$

It remains to show that f is bounded.

By assumption, there exists a neighbourhood U of 0 s.t. $U \subset A$. Then, for any $x \in U$,

$$f(x) \leq p_A(x) \leq 1.$$

hence f is bounded.

Next, consider the case when $\text{int } A \neq \emptyset$ and $B = \{b\}$. Let $a_0 \in \text{int } A$. Then by the previous case applied to $A - a_0$ and $B - a_0 = \{b - a_0\}$ there exist $f \in X^*$ and c_0 s.t. $f \neq 0$ and $f(a - a_0) \leq c_0 \leq f(b - a_0) \quad \forall a \in A$. Then for $c = c_0 + f(a_0)$ we have $f(a) \leq c \leq f(b) \quad \forall a \in A$.

Finally, consider the general case. Assume $\text{int } A \neq \emptyset$. Consider the set $C = A - B = \{a - b \mid a \in A, b \in B\}$. Then C is convex, $\text{int } C \neq \emptyset$ and $0 \notin C$. By the previous case there exists $f \in X^*$, $f \neq 0$, s.t. $f(a - b) \leq 0 = f(0) \quad \forall a \in A \quad \forall b \in B$. It follows that $\sup_{a \in A} f(a) \leq \inf_{b \in B} f(b)$, so there exists $c \in \mathbb{R}$ s.t. $f(a) \leq c \leq f(b) \quad \forall a \in A \quad \forall b \in B$. $\square$

II. Open mapping theorem

Theorem (the open mapping, or the Banach-Schauder, theorem)

Let X, Y be Banach spaces, $T: X \rightarrow Y$ be a surjective bounded linear map. Then T is open (i.e., the image of any open set is open).