

Next, consider the case when $\text{int } A \neq \emptyset$ and $B = \{b\}$. Let $a_0 \in \text{int } A$. Then by the previous case applied to $A - a_0$ and $B - a_0 = \{b - a_0\}$ there exist $f \in X^*$ and c_0 s.t. $f \neq 0$ and $f(a - a_0) \leq c_0 \leq f(b - a_0) \quad \forall a \in A$. Then for $c = c_0 + f(a_0)$ we have $f(a) \leq c \leq f(b) \quad \forall a \in A$.

Finally, consider the general case. Assume $\text{int } A \neq \emptyset$. Consider the set $C = A - B = \{a - b \mid a \in A, b \in B\}$. Then C is convex, $\text{int } C \neq \emptyset$ and $0 \notin C$. By the previous case there exists $f \in X^*$, $f \neq 0$, s.t. $f(a - b) \leq 0 = f(0) \quad \forall a \in A \quad \forall b \in B$. It follows that $\sup_{a \in A} f(a) \leq \inf_{b \in B} f(b)$, so there exists $c \in \mathbb{R}$ s.t. $f(a) \leq c \leq f(b) \quad \forall a \in A \quad \forall b \in B$. $\square$

II. Open mapping theorem

Theorem (the open mapping, or the Banach-Schauder, theorem)

Let X, Y be Banach spaces, $T: X \rightarrow Y$ be a surjective bounded linear map. Then T is open (i.e., the image of any open set is open).

②

For the proof we need the following

Theorem (the Baire category theorem)

A complete metric space cannot be written as a union of countably many nowhere-dense subsets.

(A subset A of a topological space X is called nowhere-dense if $\text{int } \bar{A} = \emptyset$.)

Proof

Assume (X, d) is a complete metric space,

$X = \bigcup_{n=1}^{\infty} A_n$, $\text{int } \bar{A}_n = \emptyset \quad \forall n$.

We will construct by induction a sequence of balls $B_{\varepsilon_n}(x_n)$ s.t.

$$\overline{B_{\varepsilon_n}(x_n)} \cap \bar{A}_n = \emptyset.$$

$\varepsilon_n \rightarrow 0$, $B_{\varepsilon_1}(x_1) \supset B_{\varepsilon_2}(x_2) \supset \dots$ and

$\frac{n=1}{\text{As } \bar{A}_1 \neq X}$, take $x \in X \setminus \bar{A}_1$ and choose $\varepsilon_1 > 0$ s.t. $B_{\varepsilon_1}(x_1) \cap \bar{A}_1 = \emptyset$.

$\frac{n-1 \rightsquigarrow n}{\text{Since } \text{int } \bar{A}_n = \emptyset, B_{\varepsilon_{n-1}}(x_{n-1}) \not\subset \bar{A}_n}$.

Take any $x_n \in B_{\varepsilon_{n-1}}(x_{n-1}) \setminus \bar{A}_n$. Then choose $\varepsilon_n > 0$ s.t. $B_{\varepsilon_n}(x_n) \subset B_{\varepsilon_{n-1}}(x_{n-1})$ and $\overline{B_{\varepsilon_n}(x_n)} \cap \bar{A}_n = \emptyset$.

The sequence $\{x_n\}_{n=1}^{\infty}$ is Cauchy:

if it. $m \geq n$ then $x_m \in B_{\varepsilon_n}(x_n)$, so $d(x_n, x_m) < \varepsilon_n$.
let $x = \lim_{n \rightarrow \infty} x_n$. Then, since $x_m \in B_{\varepsilon_n}(x_n) \quad \forall m \geq n$,
we have $x \in \overline{B_{\varepsilon_n}(x_n)} \quad \forall n$, so $x \notin \bar{A}_n \quad \forall n$.

But this contradicts $\bigcup_{n=1}^{\infty} A_n = X$. □

Proof of the open mapping theorem

Consider the open unit ball

$$B = B_1(0) = \{x \in X \mid \|x\| < 1\} \subset X.$$

Then $X = \bigcup_{n=1}^{\infty} nB$, so $Y = \bigcup_{n=1}^{\infty} nT(B)$.

By the Baire category theorem, $nT(B)$ is not nowhere dense for some $n \geq 1$.
hence $\delta + T(B)$ is not nowhere dense $\forall \delta > 0$.
It follows that there exists $y \in Y$ and $\delta > 0$

s.t. $B_{\delta}(y) \subset \frac{1}{2}T(B)$.

Then $B_{\delta}(0) = B_{\delta}(y) - y \subset \frac{1}{2}(T(B) - T(B)) \subset \overline{T(B)}$,

or equivalently,

$$B_t(0) \subset \frac{t}{\delta} \overline{T(B)} \quad \forall t > 0. \quad (*)$$

~~This means that $\forall y \in Y$ we can find $x \in X$ s.t. $\|x\| < \frac{\|y\|}{\delta}$ and Tx is arbitrarily close to y .
We claim that $B_{\delta}(0) \subset T(B)$, or in other words, $\forall y \in Y$ we claim that $B_{\delta}(0) \subset T(B)$,
or equivalently,~~

$$B_t(0) \subset \frac{t}{\delta} T(B) \quad \forall t > 0.$$

Indeed, take $y \in B_{\delta}(0)$. Fix $\frac{\|y\|}{\delta} < \varepsilon_1 < 1$.
Choose ε_n for $n \geq 2$ s.t. $\sum_{n=1}^{\infty} \varepsilon_n < 1$.

Then we inductively construct $x_n \in B_{\varepsilon_n}(0)$

s.t. $\|y - \sum_{k=1}^n Tx_k\| < \varepsilon_{n+1}\delta,$

which is possible by (*).

Put $x = \sum_{n=1}^{\infty} x_n$, so $\|x\| \leq \sum_{n=1}^{\infty} \varepsilon_n < 1$.

Then $Tx = y$.

Finally, if $A \subset X$ is open and $x \in A$, choose $\varepsilon > 0$ s.t. $B_\varepsilon(x) \subset A$.

Then

$$\begin{aligned} T(B_\varepsilon(x)) &= Tx + \varepsilon T(B) \supset Tx + B_{\varepsilon S}(0) \\ &= B_{\varepsilon S}(Tx), \end{aligned}$$

hence $T(A)$ is open in the metric space.

Corollary If T is also surjective, then T^{-1} is bounded.

Theorem (the closed graph theorem)

Let X, Y be Banach spaces, $T: X \rightarrow Y$ be a linear map. Then T is bounded if and only if $\Gamma(T) = \{(x, Tx) \mid x \in X\} \subset X \times Y$ is closed.

Proof

If T is bounded, then $\Gamma(T)$ is clearly closed. Conversely, assume $\Gamma(T)$ is closed. Then $\Gamma(T)$ is a Banach space: it is a closed subspace of the Banach space $X \oplus Y$ with norm $\|(x, y)\| = \|x\| + \|y\|$.

The map $\Gamma(T) \rightarrow X, (x, Tx) \mapsto x$,