

Then $Tx = y$.

Finally, if $A \subset X$ is open and $x \in A$, choose $\varepsilon > 0$ s.t. $B_\varepsilon(x) \subset A$.

Then

$$T(B_\varepsilon(x)) = Tx + \varepsilon T(B) \supset Tx + B_{\varepsilon S}(0) = B_{\varepsilon S}(Tx),$$

hence $T(A)$ is open.

Corollary If T is also surjective, then T^{-1} is bounded.

Theorem (the closed graph theorem)

Let X, Y be Banach spaces, $T: X \rightarrow Y$ be a linear map. Then T is bounded if and only if $\Gamma(T) = \{(x, Tx) \mid x \in X\} \subset X \times Y$ is closed.

Proof

If T is bounded, then $\Gamma(T)$ is clearly closed. Conversely, assume $\Gamma(T)$ is closed. Then $\Gamma(T)$ is a Banach space: it is a closed subspace of the Banach space $X \oplus Y$ with norm $\|(x, y)\| = \|x\| + \|y\|$. The map $\Gamma(T) \rightarrow X, (x, Tx) \mapsto x$,

III. Uniform boundedness principle

Theorem (the uniform boundedness principle, or the Banach-Steinhaus theorem)

Let X, Y be Banach spaces, $\{T_i\}_{i \in \mathbb{I}}$ be a collection of bounded linear operators $X \rightarrow Y$ s.t. $\sup_i \|T_i x\| < \infty \quad \forall x \in X$.

Then $\sup_i \|T_i\| < \infty$.

Proof

Put $X_n = \{x \in X \mid \|T_i x\| \leq n \quad \forall i \in \mathbb{I}\}$.

Then $X = \bigcup_n X_n$. Hence there exists n s.t. X_n is not nowhere dense. As X_n is closed, X_n is not nowhere dense.

As X_n is closed, this means that $\exists x_0 \in X$ and $\delta > 0$ s.t. $B_\delta(x_0) \subset X_n$. Then, as $B_\delta(0) = B_\delta(x_0) - x_0$,

we have $\|T_i x\| \leq 2n \quad \forall x \in B_\delta(0) \quad \forall i \in \mathbb{I}$.

Thus $\|T_i\| \leq \frac{2n}{\delta} \quad \forall i$. □

In many applications this principle is used in the following form.

Corollary

Let X, Y, Z be Banach spaces and $B: X \times Y \rightarrow Z$ be a bilinear map which is continuous in every variable. Then $\exists C > 0$ s.t. $\|B(x, y)\| \leq C \|x\| \|y\| \quad \forall x \in X \quad \forall y \in Y$.
In particular, $B: X \times Y \rightarrow Z$ is continuous.

Proof
By assumption there exist C_x, D_y s.t.
 $\|B(x, y)\| \leq C_x \|y\| \quad \forall y, \quad \|B(x, y)\| \leq D_y \|x\| \quad \forall x.$

Consider the unit ball $S = \{y \mid \|y\| \leq 1\} \subset Y.$

For every $S \in S$ define

$$T_S = B(\cdot, y) : X \rightarrow Z.$$

Then, $\forall x \in X, \quad \|T_S x\| = \|B(x, S)\| \leq C_x.$

We also have $\|T_S\| \leq D_S.$ hence

$$C \stackrel{\text{def}}{=} \sup_{S \in S} \|T_S\| < \infty.$$

In other words, we have

$$\|B(x, S)\| \leq C \|x\| \quad \forall x \in X \quad \forall S \in S.$$

hence $\|B(x, y)\| \leq C \|x\| \cdot \|y\|.$

Finally, B is continuous, since

$$\begin{aligned} \|B(x, y) - B(x_0, y_0)\| &\leq \|B(x, y) - B(x_0, y)\| \\ &\quad + \|B(x_0, y) - B(x_0, y_0)\| \\ &\leq C \|x - x_0\| \cdot \|y\| + C \|x_0\| \|y - y_0\|. \end{aligned}$$

□

Example

Let H, K be Hilbert spaces. Assume

$T: H \rightarrow K$ is a linear operator and

there exists a map $T^*: K \rightarrow H$ s.t.

$$(Tx, y) = (x, T^*y) \quad \forall x \in H \quad \forall y \in K.$$

Then T is bounded. Indeed, we apply

$$\text{the result to } B(x, y) = (Tx, y) = (x, T^*y)$$

to get C s.t. $\|(Tx, y)\| \leq C \|x\| \cdot \|y\|$

Example (2)
Consider $L^2(0,1)$ and the orthonormal basis

$\{e_n\}_{n \in \mathbb{Z}}$, $e_n(x) = e^{2\pi i n x}$
Recall that the Fourier coefficients of $f \in L^2(0,1)$ are

$$c_n(f) = (f, e_n) = \int_0^1 f(x) \overline{e_n(x)} dx,$$
and $f = \sum_{n \in \mathbb{Z}} c_n(f) e_n$ in $L^2(0,1)$ $\forall f \in L^2(0,1)$.

Let us show that even if $f \in C[0,1]$, $f(0) = f(1)$, we cannot expect pointwise convergence in general. More precisely:

Proposition
 $\forall x_0 \in [0,1] \exists f \in C[0,1], f(0) = f(1),$
s.t. $\left\{ \sum_{n=-N}^N c_n(f) e_n(x_0) \right\}_{N=1}^{\infty}$ is not convergent.

Proof
To simplify notation ~~we~~ assume $x_0 = 0$.
(For arbitrary x_0 we can then consider $f(x+x_0)$.)

Define a linear functional

$$\varphi_N : \text{defn } \{ f \in C[0,1] \mid f(0) = f(1) \} \rightarrow \mathbb{C}$$

$$\varphi_N(f) = \sum_{n=-N}^N c_n(f) e_n(0) = \int_0^1 f(x) \underbrace{\sum_{n=-N}^N e^{2\pi i n x}}_{g_N(x)} dx$$

It is not difficult to show (we will return to this later) that

$$\|\varphi_N\| = \|g_N\|_1 = \int_0^1 |g_N(x)| dx.$$

$$= \int_0^1 \left| \frac{\sin 2\pi(\nu + \frac{1}{2})x}{\sin 2\pi \frac{x}{2}} \right| dx$$

$$\geq \frac{1}{\pi} \int_0^1 \frac{|\sin 2\pi(\nu + \frac{1}{2})x|}{x} dx$$

$$= \frac{1}{\sqrt{\pi}} \int_0^{2\pi(n+\frac{1}{2})} \frac{|\sin x|}{x} dx \xrightarrow{n \rightarrow \infty} +\infty \text{ (exercise)}$$

By the uniform bounded principle there exists f s.t. $\{\psi_n(f)\}_{n=1}^\infty$ is unbounded. $\square$

Fact (Carleson (Abel Prize 2006))
For any $f \in L^2(0,1)$, $\sum_{n=-N}^N c_n(t) e_n(x) \xrightarrow[N \rightarrow \infty]{} f(x)$
for a.e. $x \in (0,1)$.

Fact (Kolmogorov)
There exists $f \in L^1(0,1)$ s.t. $\{\sum_{n=-N}^N c_n(t) e_n(x)\}_{N=1}^\infty$
diverges is not convergent for every $x \in [0,1]$.

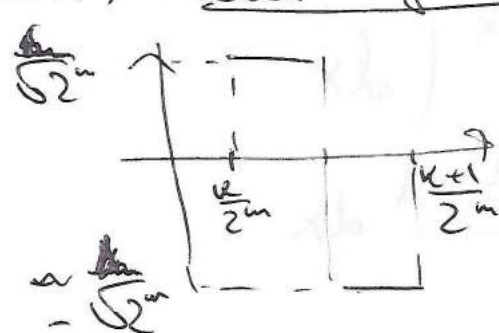
Schauder basis

Def
Schauder basis in a Banach space X is a sequence $\{e_n\}_{n=1}^\infty \subset X$ s.t. every element $x \in X$ can be uniquely written as
 $x = \sum_n \alpha_n e_n$ for scalars α_n .

Example
(i) Orthonormal basis in a separable Hilbert Space.

(ii) standard basis in ℓ_p , $1 \leq p < \infty$.

(iii) Haar system $\{x_n\}_{n=0}^\infty$ ~~approximate~~ : $x_n \equiv 1$,



$$x_n, k = n + 2^m, 0 \leq k < 2^m$$