

Schauder basis

Def A Schauder basis in a Banach space X is a sequence $\{e_n\}_{n=1}^{\infty} \subset X$ s.t. every element $x \in X$ can be uniquely written as

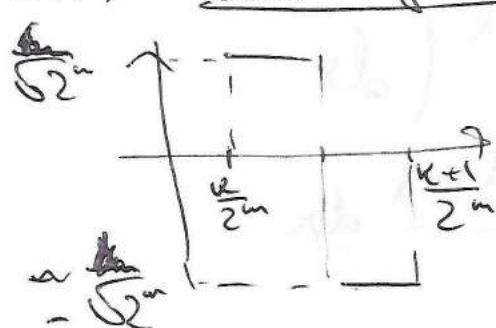
$$x = \sum_n \alpha_n e_n \quad \text{for scalars } \alpha_n.$$

Example

(i) Orthonormal basis in a separable Hilbert space.

(ii) Standard basis in ℓ_p , $1 \leq p < \infty$.

(iii) Haar system $\{x_n\}_{n=0}^{\infty}$ ~~in ℓ_1~~ $\|x_n\|_1 = 1$,



$$x_n, \|x_n\|_1 = 1, 0 \leq n < 2^m$$

It is a Schauder basis in $L^p(0,1)$ (3)
 for $1 \leq p < \infty$, and for any $f \in C[0,1]$
 the corresponding Fourier series $\sum_{n=0}^{\infty} a_n X_n$
 converges to f uniformly on $[0,1]$.

Theorem

Let X be a Banach space, $\{e_n\}_{n=1}^{\infty}$ is
 a linearly independent system of vectors
 s.t. $\text{span } \{e_n\}_n$ is dense in X . Then
 $\{e_n\}_{n=1}^{\infty}$ is a Schauder basis if and only if
 $\exists C > 0$ s.t.

$$\left\| \sum_{k=1}^n \alpha_k e_k \right\| \leq C \left\| \sum_{k=1}^m \alpha_k e_k \right\|$$

$\forall m \geq n \quad \forall \alpha_1, \dots, \alpha_m$.

Proof

⊆ Consider the dense subspace

$$X_0 = \text{span } \{e_n\}_n \subset X,$$

and linear operators $S_n: X_0 \rightarrow \text{span } \{e_k\}_{k=1}^n$,

$$S_n(\alpha_1 e_1 + \dots + \alpha_m e_m) = \alpha_1 e_1 + \dots + \alpha_n e_n$$

for $m \geq n$. By assumption, $\|S_n\| \leq C$,
 so S_n extends ~~linearly~~ by continuity
 to a linear operator $S_n: X \rightarrow \text{span } \{e_k\}_{k=1}^n$,
 which we continue to denote by S_n .

We have $S_n x \xrightarrow{n \rightarrow \infty} x \quad \forall x \in X_0$ (in fact, we see

that $S_n x = x$ for $n \gg 0$), hence $S_n x \xrightarrow{n \rightarrow \infty} x \quad \forall x \in X$.

⊇ AS $S_n S_m = S_n$ for $n \leq m$, for
 every x we have $S_n x = \sum_{k=1}^n \alpha_k e_k$ with α_k

dependency only on $k \leq n$, but not on n . Thus

$$x = \sum_{k=1}^{\infty} d_k e_k.$$

If we have another decomposition

$$x = \sum_{k=1}^{\infty} \tilde{d}_k e_k,$$

$$\text{then } S_n x = \lim_{n \rightarrow \infty} S_n \left(\sum_{k=1}^n \tilde{d}_k e_k \right) \\ = \sum_{k=1}^n \tilde{d}_k e_k,$$

$$\text{so } \tilde{d}_k = d_k \quad \forall k.$$

($\Rightarrow$) All bounded linear maps

Schauder basis

Theorem. A sequence $\{e_k\}_{k=1}^{\infty}$ in a Banach space X is a Schauder basis if and only if it is linear independent, spans a dense subspace of X and

$$(1) \quad \left\| \sum_{k=1}^n \alpha_k e_k \right\| \leq C \left\| \sum_{k=1}^m \alpha_k e_k \right\|$$

for some $C > 0$ and all $m \geq n$ and scalars $\alpha_1, \dots, \alpha_m$.

Proof. The implication $\Leftarrow$ is proved above.

In order to prove the implication $\Rightarrow$, define operators $S_n: X \rightarrow X$ by

$$S_n x = \sum_{k=1}^n \alpha_k e_k, \quad \text{if } x = \sum_{k=1}^{\infty} \alpha_k e_k.$$

Introduce a new norm $\|\cdot\|'$ on X by

$$\|x\|' = \sup_n \|S_n x\|.$$

Since $S_n x \rightarrow x$ as $n \rightarrow \infty$, this is indeed a well-defined norm and $\|x\| \leq \|x\|'$. Complete $(X, \|\cdot\|')$ to a Banach space X' . The identity map $X \rightarrow X$ extends by continuity to a contraction $T: X' \rightarrow X$.

Observe that condition (1) is satisfied for the norm $\|\cdot\|'$ and $C = 1$. The span of the elements e_k is dense in X' , since X is dense and for every $x \in X$ we have

$$\|x - S_n x\|' = \sup_{m > n} \|S_m x - S_n x\| \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

By the implication $\Leftarrow$ that we have already proved, it follows that $\{e_k\}_{k=1}^{\infty}$ is a Schauder basis in X' .

Assume now that $Tx' = 0$ for some $x' \in X'$. Write x' as $\sum_{k=1}^{\infty} \alpha_k e_k$ (with convergence in X'). Since T is bounded and $Te_k = e_k$, we then get

$$0 = Tx' = \sum_{n=1}^{\infty} \alpha_n e_n \quad \text{in } X.$$

Since $\{e_k\}_k$ is a Schauder basis in X , this is possible only if $\alpha_k = 0$ for all k . Hence $x' = 0$ and therefore T is injective. Since T is also surjective, T is a linear isomorphism. In other words, X is complete with respect to $\|\cdot\|'$.

As $T = \text{id}: (X, \|\cdot\|') \rightarrow (X, \|\cdot\|)$ is bounded, by the open mapping theorem we conclude that $T^{-1} = \text{id}: (X, \|\cdot\|) \rightarrow (X, \|\cdot\|')$ is bounded as well. Hence

$$\|S_n x\| \leq \|T^{-1}\| \|x\|$$

for any $x \in X$ and $n \geq 1$. This gives (1) for $x = \sum_{k=1}^m \alpha_k e_k$ and $C = \|T^{-1}\|$. □

It can be shown (Enflo) that there
exist a separable Banach Space without
a Schauder basis. In fact, such
a space can be found ~~as a subspace~~^{is a}
of ℓ_p for $1 \leq p < \infty$, $p \neq 2$
(Davie for $p > 2$, Szankowski for $1 \leq p < 2$).