

Recollection:

X a set, $\mathcal{B}$ a σ -algebra ($\emptyset \in \mathcal{B}$, complements, countable unions), μ a measure on $(X, \mathcal{B})$:
 $\mu: \mathcal{B} \rightarrow [0, +\infty]$, $\mu(\emptyset) = 0$,
 $\mu(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mu(A_n)$ if $A_n \cap A_m = \emptyset$ for $n \neq m$.

Example

Lebesgue measure on $\mathbb{R}$:
First define on Borel sets as $\lambda(a, b) = b - a$.
Then add to the σ -algebra $\mathcal{B}$ of Borel sets all Lebesgue measurable sets $A \subset \mathbb{R}$ s.t. $A \in \mathcal{B}$, $\lambda(A) < \infty$. The σ -algebra we get is the σ -algebra of Lebesgue measurable sets.
Equivalently, $A \subset \mathbb{R}$ is Lebesgue measurable if and only if $\exists B_0, B_1$ Borel s.t.
 $B_0 \subset A \subset B_1$
s.t. $\lambda(B_1 \setminus B_0) = 0$. Moreover, we can choose B_0 to be F_σ and B_1 to be G_δ . The $\lambda(A) = \lambda(B_0) = \lambda(B_1)$.

Given a measure space $(X, \mathcal{B}, \mu)$,
the Lebesgue integral $(\int_X f d\mu)$ is first
defined for all measurable nonnegative
functions $X \rightarrow [0, +\infty]$. It is a map
 $f \mapsto \int_X f d\mu \in [0, +\infty]$ uniquely determined by:
(i) $\int_X \chi_A d\mu = \mu(A) \quad \forall A \in \mathcal{B}$;
(ii) $\int_X (\alpha f + \beta g) d\mu = \alpha \int_X f d\mu + \beta \int_X g d\mu$
 $\forall \alpha, \beta \geq 0$ if measurable $f, g: X \rightarrow [0, +\infty]$.

(iii) If $f, f_n: X \rightarrow [0, +\infty]$ are measurable,
 $f_n \leq f$ and $f_n(x) \rightarrow f(x)$ for μ -a.e. $x \in X$,
 then $\int_X f_n d\mu \rightarrow \int_X f d\mu$.

(A form of the dominated convergence theorem)
 (The ^{of} ~~theorem~~ ^{of} the monotone convergence theorem: $f_n = \bigwedge_{m \geq n} f_m$)
 A measurable function $f: X \rightarrow \mathbb{C}$ is
 called integrable if $\int_X |f| d\mu < +\infty$.

Then the integral is the unique linear
 functional on the space of integrable
 functions extending $f \mapsto \int_X f d\mu$ for
 nonnegative f .

Dominated convergence theorem

If f_n are integrable, $f_n(x) \rightarrow f(x)$
 for μ -a.e. x , and $|f_n| \leq g$
 for some integrable g , then f is
 integrable and
 $\int_X f_n d\mu \rightarrow \int_X f d\mu$.

Example

If f is a Riemann integrable function
 on $[0, b]$, then f is Lebesgue integrable
 and $\int_a^b f(x) dx = \int_{[0, b]} f d\lambda$.

For $1 \leq p < \infty$ a measurable function $f: X \rightarrow \mathbb{C}$
 is called p -summable if $\int_X |f|^p d\mu < \infty$
 (so $|f|^p$ is integrable)

For $1 \leq p < \infty$ we define $L^p(X, \mathcal{B}, \mu) = L^p(X, d\mu) \stackrel{\text{def}}{=} \frac{\text{Space of } p\text{-summable fct.}}{\text{Subspace of } f \text{ s.t. } f=0 \mu\text{-a.e.}}$

We usually do not distinguish between elements of $L^p(X, d\mu)$ and p -summable functions.

$L^p(X, d\mu)$ is a Banach space with norm

$$\|f\|_p = \left(\int_X |f|^p d\mu \right)^{1/p}$$

A measurable function

Assume $(X, \mathcal{B}, \mu)$ is σ -finite
(so $X = \bigcup_{n=1}^{\infty} X_n$, $\mu(X_n) < +\infty$).

A measurable function $f: X \rightarrow \mathbb{C}$ is called essentially bounded if there exists a bounded function $\tilde{f}$ s.t. $f = \tilde{f}$ μ -a.e.

$L^\infty(X, \mathcal{B}, \mu) = L^\infty(X, \mu) \stackrel{\text{def}}{=} \frac{\text{Space of ess. bounded fct.}}{\text{Subspace of } f \text{ s.t. } f=0 \mu\text{-a.e.}}$

This is a Banach space with norm

$$\|f\|_\infty = \inf_{\tilde{f}: \tilde{f}=f \mu\text{-a.e.}} \sup_{x \in X} |\tilde{f}(x)|$$

$$= \sup_{\substack{A \subset X \\ \mu(A^c) = 0}} \sup_{x \in A} |f(x)|$$

For the proof we will need Dynkin's theorem

Recall that a collection $\mathcal{D}$ of subsets of X is a Dynkin system if $\emptyset \in \mathcal{D}$, it is closed under complements and under countable unions.

of disjoint sets.

(3)

Theorem (Dynkin)

Assume S is a collection of sets closed under finite intersections. Then the Dynkin system $D(S)$ generated by S is a σ -algebra.

Proof

It suffices to show that $D(S)$ is closed under finite intersections.

Fix $A \in D(S)$.

Consider $D_A \stackrel{\text{def}}{=} \{ B \in D(S) \mid A \cap B \in D(S) \}$.

We claim that D_A is a Dynkin system.

Indeed, clearly $\emptyset \in D_A$. If $B \in D_A$ then

$$A \cap B^c = A \setminus (A \cap B) = (A^c \cup (A \cap B)^c)^c \in D(S),$$

so $B^c \in D_A$.

Finally, if $B = \bigcup_{n=1}^{\infty} B_n$, $B_n \in D$, then

$$A \cap B = \bigcup_{n=1}^{\infty} (A \cap B_n) \in D(S), \text{ so } B \in D_A.$$

Thus the claim is proved.

It follows, that if $A \in S$ then

$S \subset D_A$, so $D_A = D(S)$. Thus

$$B \in D(S) \Rightarrow A \cap B \in D(S) \quad \forall A \in S.$$

But this means that $S \subset D_B \quad \forall B \in D(S)$,

so $D_B = D(S) \quad \forall B$.

Corollary A finite measure on $D(S)$ is determined by its values on S and X . □