

Let  $(X_1, \mathcal{B}_1, \mu_1)$  and  $(X_2, \mathcal{B}_2, \mu_2)$  be two measure spaces.

Consider the space  $X_1 \times X_2$ .

The product  $\sigma$ -algebra is the smallest  $\sigma$ -algebra of subset of  $X_1 \times X_2$  containing all sets of the form  $A_1 \times A_2$  for  $A_1 \in \mathcal{B}_1, A_2 \in \mathcal{B}_2$ . It is denoted by  $\mathcal{B}_1 \times \mathcal{B}_2$  or  $\mathcal{B}_1 \otimes \mathcal{B}_2$ .

### Theorem

Assume  $(X_1, \mathcal{B}_1, \mu_1)$  and  $(X_2, \mathcal{B}_2, \mu_2)$  are  $\sigma$ -finite, and  $f: X \rightarrow [0, +\infty]$  is a  $\mathcal{B}_1 \times \mathcal{B}_2$ -measurable function.

Then ~~the following~~ the following functions are measurable:

$$(X, \mathcal{B}_1) \rightarrow [0, +\infty], x_1 \mapsto f(x_1, x_2) \quad (\forall x_2 \in X_2),$$

$$(X_2, \mathcal{B}_2) \rightarrow [0, +\infty], x_2 \mapsto f(x_1, x_2) \quad (\forall x_1 \in X_1),$$

$$(X_1, \mathcal{B}_1) \rightarrow [0, +\infty], x_1 \mapsto \int_{X_2} f(x_1, x_2) d\mu(x_2),$$

$$(X_2, \mathcal{B}_2) \rightarrow [0, +\infty], x_2 \mapsto \int_{X_1} f(x_1, x_2) d\mu(x_1),$$

and

$$\int_{X_1} d\mu_1(x_1) \int_{X_2} d\mu_2(x_2) f(x_1, x_2) = \int_{X_2} d\mu_2(x_2) \int_{X_1} d\mu_1(x_1) f(x_1, x_2)$$

For the proof we will need Dynkin's theorem

Recall that a collection  $\mathcal{D}$  of subsets of  $X$  is a Dynkin system if  $\emptyset \in \mathcal{D}$ , it is closed under complements and under countable unions.

of disjoint sets. (3)

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Proof of theorem (on product measures)  
let us first consider the case when  
 $\mu_1$  and  $\mu_2$  are finite and  $X = X_A$ .

Consider the  $\sigma$ -D  $\mathcal{D} \subset \mathcal{B}_1 \times \mathcal{B}_2$  of measurable sets  $A$  s.t. the theorem holds for  $\chi_A$ . Then  $\mathcal{D}$  contains all sets of the form  $A_1 \times A_2$ . Since such sets generate  $\mathcal{B}_1 \times \mathcal{B}_2$  as a  $\sigma$ -algebra, to prove that  $\mathcal{D} = \mathcal{B}_1 \times \mathcal{B}_2$  it suffices to show that  $\mathcal{D}$  is a Dynkin system.  $\mathcal{D}$  is closed under complements, (here it is important as  $\chi_{A^c} = 1 - \chi_A$ )

that  $\mu_1, \mu_2$  are finite!)  
 Finally,  $\mathcal{D}$  is closed under disjoint countable unions by additivity of integrals and the monotone convergence theorem.

We thus conclude that the theorem is true for  $f = \chi_A$  with  $A \subset Y_1 \times Y_2$ ,  $\mu_1(Y_1), \mu_2(Y_2) < +\infty$ .

The general case follows from this by additivity and the monotone convergence theorem. □

## Theorem

Let  $(X_1, \mathcal{B}_1, \mu_1)$  and  $(X_2, \mathcal{B}_2, \mu_2)$  be  $\sigma$ -finite measure spaces. Then

(i) (Tonelli's theorem) for any  $(\mathcal{B}_1 \times \mathcal{B}_2)$ -measurable function  $f: X_1 \times X_2 \rightarrow [0, +\infty]$  we have

$$(*) \int_{X_1 \times X_2} f d(\mu_1 \times \mu_2) = \int_{X_1} \left( \int_{X_2} f d\mu_2 \right) d\mu_1 = \int_{X_2} \left( \int_{X_1} f d\mu_1 \right) d\mu_2;$$

(ii) (Fubini's theorem) for any  $(\mathcal{B}_1 \times \mathcal{B}_2)$ -measurable function  $f: X_1 \times X_2 \rightarrow \mathbb{C}$ , if one of the integrals

$$(**) \int_{X_1 \times X_2} |f| d(\mu_1 \times \mu_2), \int_{X_1} \left( \int_{X_2} |f| d\mu_2 \right) d\mu_1, \int_{X_2} \left( \int_{X_1} |f| d\mu_1 \right) d\mu_2$$

is finite, then the other two are finite as well, so  $f$  is  $(\mu_1 \times \mu_2)$ -integrable, and

(\*) holds.

(~~\*)~~  $\int_{X_1} f(\cdot, x_2) d\mu_1 \in L^1(\mu_2)$   $\mu_2$ -a.e.,  
~~separable~~  $\left( \int_{X_1} f(x_1, \cdot) d\mu_1 \in L^1(\mu_2), \dots \right)$

## Proof

(i) This is true for  $f = \chi_A$  by the construction of  $\mu_1 \times \mu_2$ . The general case

follows from this by the monotone convergence theorem.

(ii) The first statement follows from (i), as all three integrals coincide. To prove the second part, we can consider separately  $(\operatorname{Re} f)_\pm$  and  $(\operatorname{Im} f)_\pm$ .

Thus we may assume that  $f \geq 0$ .

By part (i) we have

$$\int_{X_1} \left( \int_{X_2} f(x_1, x_2) d\mu_2(x_2) \right) d\mu_1(x_1) = \int_{X_1 \times X_2} f d(\mu_1 \times \mu_2) < \infty.$$

Thus the function  $x_1 \mapsto \int_{X_2} f(x_1, x_2) d\mu_2(x_2)$  is integrable. In particular, it is finite  $\mu_1$ -a.e., that is, for  $\mu_1$ -a.e.  $x_1$ , the function  $(x_2 \mapsto f(x_1, x_2))$  is  $\mu_2$ -integrable.

Similarly for  $\int_{X_2} \left( \int_{X_1} f d\mu_1 \right) d\mu_2$ . □