

The Radon-Nikodym theorem

Consider a measurable space $(X, \mathcal{B})$.

Def

For measures μ and ν on $(X, \mathcal{B})$, ν is called absolutely continuous with respect to μ if

$$A \in \mathcal{B}, \mu(A) = 0 \Rightarrow \nu(A) = 0.$$

We then write $\nu \ll \mu$.

Example

Let μ be a measure on $(X, \mathcal{B})$ and $f \in L^1(X, d\mu)$. Then the measure

$$\nu(A) = \int_A f d\mu$$

is abs. s.t. $\nu \ll \mu$. We write $d\nu = f d\mu$.

Def Proposition

Let μ, ν be measures on $(X, \mathcal{B})$, $\nu(X) < \infty$. Then $\nu \ll \mu$ if and only if $\forall \varepsilon > 0 \exists \delta > 0$ s.t. if $\mu(A) < \delta$ for some $A \in \mathcal{B}$, then $\nu(A) < \varepsilon$.

Proof

($\Leftarrow$) Obvious

($\Rightarrow$) Assume $\nu \ll \mu$ the statement is not true.

So $\exists \varepsilon > 0$ s.t. $\forall \delta > 0$ there exists $A \in \mathcal{B}$ with $\mu(A) < \delta$ but $\nu(A) \geq \varepsilon$.

Take $\delta = \frac{1}{2^n}$ and choose the corresponding A_n .

Put $A = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k$.

Then $\mu(A) = \lim_{n \rightarrow \infty} \mu\left(\bigcup_{k=n}^{\infty} A_k\right) \leq \lim_{n \rightarrow \infty} \sum_{k=n}^{\infty} \frac{1}{2^k} = 0$,

while $\nu(A) = \lim_{n \rightarrow \infty} \nu\left(\bigcup_{k=n}^{\infty} A_k\right) \geq \varepsilon$;

which is a contradiction. $\square$

Exercise

Show that the result is not true without the assumption $\nu(X) < \infty$.

Proposition

Let μ, ν be σ -finite measures on $(X, \mathcal{B})$. Then there exists $N \in \mathcal{B}$ and a $\mathcal{B}$ -measurable function $f: X \rightarrow [0, +\infty)$ s.t.

$$\mu(N) = 0 \text{ and } \nu(A) = \nu(A \cap N) + \int_A f d\mu.$$

Proof

Assume first that the measures μ and ν are finite. Consider the measure $\eta = \mu + \nu$.

Define a linear functional φ on

$$L^2(X, d\eta) \text{ by}$$

$$\varphi(g) = \int_X g d\nu.$$

$$\begin{aligned} |\varphi(g)| &= \left| \int_X g d\nu \right| \leq \left(\int_X |g|^2 d\nu \right)^{1/2} \left(\int_X 1 d\nu \right)^{1/2} \\ &\leq \nu(X)^{1/2} \left(\int_X |g|^2 d\eta \right)^{1/2} \end{aligned}$$

Hence there exists $h \in L^2(X, d\eta)$ s.t.

$$(*) \quad \int_X g d\nu = \int_X g h d\eta \quad \forall g \in L^2(X, d\eta).$$

Thus $\nu(A) = \int_A h d\eta \quad \forall A \in \mathcal{B}$. In particular,

$h \geq 0$ η -a.e. Since $v(A) \leq \eta(A) \quad \forall A \in \mathcal{B}$,
we also clearly have $h \leq 1$ η -a.e.

Observe also that by the monotone convergence theorem ~~(*) holds for all measurable $g \geq 0$~~ .
In ~~the case of $g = \chi_A$~~ , we have

$$\int_X g \, d\nu = \int_X g h \, d\nu + \int_X g h \, d\mu, \text{ for } g = \chi_A,$$

whence

$$\int g(1-h) \, d\nu = \int_X g h \, d\mu, \quad (*)$$

for $g = \chi_A$, hence for all $g \geq 0$ by the monotone convergence theorem.
Put $N = \{x \mid h(x) = 1\}$ and $f = \frac{h}{1-h} \chi_{N^c}$.

Letting $g = \chi_N$ in ~~(*)~~ we get $0 = \mu(N)$,

while letting $g = \frac{\chi_{N^c} \chi_A}{1-h}$ we get

$$\begin{aligned} \nu(N^c \cap A) &= \int_X \frac{\chi_N \chi_A h}{1-h} \, d\mu = \int_X \chi_A f \, d\mu \\ &= \int_A f \, d\mu, \end{aligned}$$

so that

$$\nu(A) = \nu(N \cap A) + \nu(N^c \cap A) = \nu(N \cap A) + \int_A f \, d\mu.$$

Consider now the general case.

We can write $X = \bigcup_n X_n = \bigcup_n Y_n$ s.t.

$\mu(X_n), \nu(Y_n) < \infty$. Applying the first part of the proof to $X_n \cap Y_m$, we get $N_{nm} \subset X_n \cap Y_m$ and measurable

$f_{nm}: X_n \cap Y_m \rightarrow [0, +\infty]$ s.t. $\mu(N_{nm}) = 0$,

$$\nu(A \cap X_n \cap Y_m) = \nu(A \cap N_{nm}) + \int_{A \cap X_n \cap Y_m} f_{nm} \, d\mu.$$

Letting $N = \bigcup_{n,m} N_{nm}$ and $f = \sum_{n,m} f_{nm} \chi_{(X_n \cap Y_m)}$,
we get the result. □

Theorem

Let $\nu \ll \mu$ be σ -finite measures on $(X, \mathcal{B})$.
Then there exists a unique, modulo sets of μ -measure 0, measurable function $f: X \rightarrow [0, +\infty)$ s.t. $d\nu = f d\mu$.

The function f is called the Radon-Nikodym derivative of ν w.r.t. μ and is denoted by $\frac{d\nu}{d\mu}$.

Proof

Existence follows from the proposition, since $\nu(N) = 0$ by assumption.

To prove uniqueness, it suffices to consider the case $\nu(X) < \infty$.

Assume we have two such functions f_1, f_2 .

Then $f_1, f_2 \in L^1(X, d\mu)$ and $\int_A (f_1 - f_2) d\mu = 0$

$\forall A \in \mathcal{B}$. By considering the sets where

$f_1 - f_2 \geq 0$ and $f_1 - f_2 \leq 0$, we get $f_1 - f_2 = 0$

μ -a.e. □