

Def
Let ν, μ be measures on $(X, \mathcal{B})$.
If ν is called singular w.r.t. μ if
there exist $N \in \mathcal{B}$ s.t. $\nu(N^c) = 0$ and $\mu(N) = 0$.
We then write $\nu \perp \mu$.

Theorem (Lebesgue decomposition)
Let μ, ν be σ -finite measures on $(X, \mathcal{B})$.
Then there exists a unique decomposition

$\nu = \nu_a + \nu_s$, where $\nu_a \ll \mu$ and $\nu_s \perp \mu$. (3)

Proof

let N and f be as in the proposition, so $\mu(N) = 0$ and $\nu(A) = \nu(N \cap A) + \int_A f d\mu$.

Then we can put $\nu_a(A) = \int_A f d\mu$, $\nu_s(A) = \nu(A \setminus N)$.

If we have another decomposition

$$\nu = \tilde{\nu}_a + \tilde{\nu}_s$$

with $\tilde{\nu}_a \ll \mu$, $\tilde{\nu}_s \perp \mu$, $\tilde{\nu}_s(\tilde{N}^c) = 0$, $\mu(\tilde{N}) = 0$,

then for $A \subset N \cup \tilde{N}$ we have

$$\nu_a(A) = \tilde{\nu}_a(A) = 0 \quad (\text{as } \mu(N \cup \tilde{N}) = 0), \text{ so}$$

$$\nu_s(A) = \nu(A) = \tilde{\nu}_s(A),$$

while for $A \subset (N \cup \tilde{N})^c$ we have

$$\nu_s(A) = \tilde{\nu}_s(A) = 0, \text{ so } \nu_a(A) = \tilde{\nu}_a(A).$$

hence $\nu_s = \tilde{\nu}_s$ and $\nu_a = \tilde{\nu}_a$. $\square$

Let $(X_1, \mathcal{B}_1, \mu)$ be a measure space,

$(X_2, \mathcal{B}_2)$ a measurable space, and

$f: X_1 \rightarrow X_2$ a measurable map

(so $f^{-1}(A) \in \mathcal{B}_1$, $\forall A \in \mathcal{B}_2$).

Def

The image, or the pushforward, of the measure μ under f is the measure $f_*\mu$ on $(X_2, \mathcal{B}_2)$ defined by

$$(f_*\mu)(A) = \mu(f^{-1}(A)).$$

For any measurable $g: X_2 \rightarrow [0, +\infty]$ we have

$$\int_{X_2} g d(f_*\mu) = \int_{X_1} g \circ f d\mu.$$

Theorem

Let $\Omega_1, \Omega_2 \subset \mathbb{R}^n$ be open sets and

$F: \Omega_1 \rightarrow \Omega_2$ be a C^1 -diffeomorphism.

Consider $\mu_1 = \lambda_n|_{\Omega_1}$ and $\mu_2 = \lambda_n|_{\Omega_2}$.

F is Lebesgue measurable and

Then $\forall \frac{d(F_*\mu_1)}{d\mu_2} = |\det T_F| = |\det T_F| \circ F^{-1}$,

where T_F is the Jacobian of F .

Proof

We claim that for any Borel $f: \Omega_2 \rightarrow [0, \infty]$ we have

$$\int_{\Omega_1} f \circ F |\det T_F| d\lambda_n = \int_{\Omega_2} f d\lambda_n. \quad (*)$$

We will show this in several steps:

1) True for $f = \chi_{[0, b_1] \times \dots \times [0, b_n]}$.

This is either proved directly (using that F^{-1} maps a small parallelepiped P into something close to a parallelepiped of volume $|\det T_F(F^{-1}(x_0))| \cdot \text{Vol}(P)$), or by using that

then (*) is known for $f \in C_c(\Omega_2)$

and then using the monotone convergence theorem.

2) True for $f = \chi_P$ for any rectangular parallelepiped of any type, by the monotone convergence theorem.

3) True for $f = \chi_A$ for any Borel ACP, for $P \subset \mathbb{R}^n$ any rectangular parallelepiped, since the class of ACP for which (*) holds forms a Dynkin system, and parallelepipeds are closed under finite intersections.

- 4) true for $f = \chi_A$, with A contained in a finite number of parallelepipeds. (9)
 5) True ^{for $f = \chi_A$} for any Borel $A \subset \Omega_2$.
 6) true for any Borel $f: \Omega_2 \rightarrow [0, +\infty)$.

This implies, that for any Borel $A \subset \Omega_2$, we have $\lambda_n(A) \geq 0 \Leftrightarrow \lambda_n(F(A)) = 0$.

Thus F is Lebesgue measurable.

Since any Lebesgue measurable function coincides a.e. with a Borel function, we also conclude that (*) holds for all Lebesgue measurable $f: \Omega_2 \rightarrow [0, +\infty]$. We can also rewrite (*) as

$$\int_{\Omega_2} f \circ F d\lambda_n = \int_{\Omega_2} f |\det J_F| d\lambda_n$$

$$\int_{\Omega_2} f d(F_*\lambda_n)$$

For any Lebesgue measurable $f: \Omega_2 \rightarrow [0, +\infty)$, but this exactly means that $\frac{d(F_*\lambda_n)}{d\lambda_n} = |\det J_F|$.

□