

Regularity of Borel measures

Def Assume X is a metric space and μ is a Borel measure on X . Then μ is called outer regular, if

$$\mu(A) = \inf_{\substack{A \subset U \\ U \text{ is open}}} \mu(U) \quad \forall \text{ Borel } A \subset X,$$

and it is called inner regular, if

$$\mu(A) = \sup_{\substack{K \subset A \\ K \text{ is compact}}} \mu(K) \quad \forall \text{ Borel } A \subset X,$$

and μ is called regular if it is both outer and inner regular.

Warning

Sometimes by inner regularity it is meant that $\mu(A) = \sup_{\substack{F \subset A \\ F \text{ closed}}} \mu(F)$, while our definition of regularity is called tightness.

Theorem

Assume X is a metric space and μ is a Borel measure on X s.t. the measure of every ball is finite. Then μ is σ -finite and for any Borel set $A \subset X$ and any $\epsilon > 0$ there exist an open set U and a closed set F s.t.

$$F \subset A \subset U \quad \text{and} \quad \mu(U \setminus F) < \epsilon. \quad (*)$$

In particular, μ is outer regular. If, furthermore, X is complete and separable, then μ is also inner regular.

Proof

σ -finiteness is clear.

Turning to (b), assume first that μ is finite.
Then (b) $\forall \epsilon > 0$ is equivalent to

$$\sup_{F \subset A} \mu(F) = \mu(A) = \sup_{A \subset U} \mu(U). \quad (**)$$

The set of sets satisfying (**) forms a σ -algebra containing all closed sets (as they are G_δ -sets), hence it coincides with the σ -algebra of Borel sets. Assume now that μ is infinite. Choose $x_0 \in X$.

Put $A_n = \{x \in X \mid n-1 \leq d(x, x_0) < n\}$

By the finite measure case ~~there exist~~ (applied to $A_n \subset B_{n+1/2}(x_0)$) there exist sets F_n and U_n that are closed, resp. open, $\mu(B_{n+1/2}(x_0) \setminus F_n) < \frac{\epsilon}{2^n}$ and $F_n \subset A_n \subset U_n$. Then

F_n is closed in X , as $F_n \subset B_n(x_0)$. Consider $F = \bigcup_{n=1}^{\infty} F_n$, $U = \bigcup_{n=1}^{\infty} U_n$. Then F is closed, U is open and $\mu(U \setminus F) < \epsilon$.

In particular, $\mu(A) \leq \mu(U) + \epsilon$, so μ is outer regular.

Assume now that X is complete and separable. For any Borel set A we have

$$\mu(A) = \sup_{F \subset A} \mu(F) = \sup_{F \text{ closed}} \sup_{F \subset A} \mu(F).$$

It follows that in proving inner regularity we may assume that A is closed and μ is finite.

Then A is itself separable and complete.

Therefore it suffices to consider $A = X$.

Fix $\epsilon > 0$. For every n we can cover X by countably many open balls $U_{n,k}$, $k \geq 1, 2, \dots$, of radius 2^{-n} . Then, for every n , we

can choose m_n s.t. $\mu(X \setminus W_n) < \varepsilon 2^{-n}$ for $W_n = \bigcup_{k=1}^{m_n} U_{n,k}$ (2)
 we have $\mu(X \setminus W_n) < \varepsilon 2^{-n}$. Consider the
 set $W = \bigcap_{n=1}^{\infty} W_n$. Then $\mu(X \setminus W) < \varepsilon$.
 For every n the set W is contained in W_n ,
 so it can be covered by finitely many balls of radius 2^{-n} .
 Therefore $\forall \delta > 0$ there exists a finite δ -net in W .
 Hence $K = \overline{W}$ is compact and $\mu(X \setminus K) < \varepsilon$. $\square$

Proposition

Let X be a metric space and μ be a Borel measure on X s.t. $\mu(B) < \infty$ for every ball $B \subset X$. Then $\forall 1 \leq p < \infty$ the space $\mathcal{L}$ of bounded continuous functions with bounded support is dense in $L^p(X, \mu)$.

Proof

It suffices to show that $\chi_A \in \overline{\mathcal{L}} \subset L^p(X, \mu)$ $\forall$ Borel A with $\mu(A) < \infty$.

By outer regularity it suffices to consider open A . Then put

$$f_n(x) = \frac{\text{nd}(x, A^c)}{1 + \text{nd}(x, A^c)}.$$

Then $f_n \uparrow \chi_A$ pointwise. This gives the required approximation for bounded open A , for unbounded A approximate χ_A by $\chi_{A \cap B_n(x_0)}$. □