

Def

A function f on $[a, b]$ is called absolutely continuous if $\forall \varepsilon > 0 \exists \delta > 0$ s.t. if

$$a \leq x_1 \leq y_1 \leq x_2 \leq y_2 \leq \dots \leq x_n \leq y_n = b$$

and $\sum_{i=1}^n (y_i - x_i) < \delta$, then

$$\sum_{i=1}^n |f(y_i) - f(x_i)| < \varepsilon.$$

Exercise

Show that if $f \in L^1(a, b)$, then $F(x) = \int_a^x f dt$ is absolutely continuous.

It is clear that absolutely continuous functions on $[a, b]$ form a vector space $AC[a, b]$ and f is absolutely continuous if and only if $\overline{f}$ is. In particular, if $f \in AC[a, b]$, then $Re f, Im f \in AC[a, b]$.

We want to decompose real absolutely continuous

functions further.

For a function f on $[a, b]$, ~~denoted by~~
the total variation of f on $[a, b]$ is defined
by

$$V_a^b(f) = \sup_{a=x_0 < \dots < x_n=b} \sum_{k=1}^n |f(x_k) - f(x_{k-1})|$$

If $V_a^b(f) < \infty$, then f is said to be
of bounded variation.

Lemma

If $f \in AC[a, b]$, then f has bounded
variation.

Proof

Fix $\varepsilon > 0$ and find $\delta > 0$ according
to the definition of AC. Then

$$V_{x_1}^{x_2}(f) \leq \varepsilon \quad \text{if} \quad |x_2 - x_1| < \delta.$$

If $x_1 < x_2 < x_3$, we always have

$$V_{x_1}^{x_3}(f) = V_{x_1}^{x_2}(f) + V_{x_2}^{x_3}(f) \quad (\text{exercise}).$$

hence, if we choose $a = x_0 < \dots < x_n = b$
s.t. $x_k - x_{k-1} < \delta$, we get

$$V_a^b(f) = \sum_{k=1}^n V_{x_{k-1}}^{x_k}(f) \leq n\varepsilon.$$

□

Lemma

Let $f \in AC[a, b]$ be real. ~~Put~~

~~$g(x) = V_a^x(f)$ and $h(x) = g(x) - f(x)$.~~
Put $g(x) = V_a^x(f)$ and $h(x) = g(x) - f(x)$.
Then g and h are ^{absolutely} continuous nondecreasing
functions.

Proof

(2)

Fix $\varepsilon > 0$ and choose $\delta > 0$ as in the definition of absolute continuity of f .

Let $a \leq x_1 \leq y_1 \leq \dots \leq x_n \leq y_n \leq b$ be s.t.

$$\sum_k (y_k - x_k) < \delta.$$

Taking any disjoint intervals (a_{kj}, b_{kj}) , $j \in X_k$, in (x_k, y_k) , the total length of the intervals (a_{kj}, b_{kj}) ($j \in X_k$, $1 \leq k \leq n$) is still $< \delta$, hence

$$\sum_k \sum_{j \in X_k} |f(b_{kj}) - f(a_{kj})| < \varepsilon.$$

This implies that

$$\sum_k V_{x_k}^{y_k}(f) \leq \varepsilon.$$

As $V_{x_k}^{y_k}(f) = V_a^{y_k}(f) - V_a^{x_k}(f)$, this shows that $g = V_a^{\cdot}(f)$ is absolutely continuous. Hence $h = g - f$ is also absolutely continuous. Clearly, g is nondecreasing. If $x < y$,

$$\begin{aligned} \text{then } V_a^y(f) &= V_a^x(f) + V_x^y(f) \geq V_a^x(f) + |f(y) - f(x)| \\ &\geq V_a^x(f) - f(y) + f(x), \end{aligned}$$

so $h(y) \geq h(x)$. Thus h is also nondecreasing. $\square$

Theorem ("fundamental theorem of calculus")

Assume $f \in AC[a, b]$. Then f is differentiable a.e. $x \in [a, b]$, $f' \in L^1(a, b)$ and

$$f(x) = f(a) + \int_a^x f' d\lambda \quad \forall x \in [a, b]. \quad \textcircled{2}$$

Proof

By the previous lemmas we may assume that f is real and nondecreasing. Consider the corresponding Lebesgue-Stieltjes measure μ , so

$$\mu((x, y]) = f(y) - f(x).$$

We claim that μ is absolutely continuous w.r.t. λ . We have to show that $\forall \epsilon > 0$

$\exists \delta > 0$ s.t. if $\lambda(A) < \delta$ then $\mu(A) < \epsilon$.

By outer regularity of λ and μ it suffices to consider open A . Any open subset of $\mathbb{R}$ is a disjoint union of at most countably many ~~disjoint~~ open intervals. It suffices to consider only disjoint unions of finitely many intervals. Let $\delta > 0$ be as in the definition of absolute continuity of f . If we have disjoint intervals (x_k, y_k) ($1 \leq k \leq n$), then of total length $< \delta$.

$$\mu\left(\bigcup_{k=1}^n (x_k, y_k)\right) = \sum_{k=1}^n (f(y_k) - f(x_k)) < \epsilon.$$

Thus μ is indeed absolutely continuous.

Let $h = \frac{df}{d\lambda} \in L^1(a, b)$. Then

$$f(x) = f(a) + \mu((a, x]) = f(a) + \int_a^x h d\lambda.$$

It follows that f' exists a.e. and

$$f'(x) = h(x).$$

□