

Complex measures

$$(\Sigma \rightarrow \mathbb{C})$$

Assume (X, Σ) is a measurable space.

Def
A map $\mu: \Sigma \rightarrow \mathbb{C}$ is called a complex measure if

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n)$$

for $A_n \in \Sigma$, $A_n \cap A_m = \emptyset$ for $n \neq m$ (the series

is assumed to converge absolutely)

(If $\nu(\Sigma) < \infty$, then ν is a finite signed measure)

Example

Assume μ is a positive measure, $f \in L^1(X, \Sigma, \mu)$.

Then $\nu = f d\mu$ is a complex measure, so

$$\nu(A) = \int_A f d\mu.$$

Def

The total variation of a complex measure ν is the map $|\nu|: \Sigma \rightarrow \mathbb{R}_+$ defined by

$$|\nu|(A) = \sup \left\{ \sum_{k=1}^n |\nu(A_k)| : A = \bigcup_{k=1}^n A_k, A_k \cap A_m = \emptyset \text{ for } k \neq m \right\}$$

(we will see soon that indeed $|\nu|(A) < \infty \forall A \in \Sigma$).

Theorem

The total variation of a complex measure is a finite positive measure on (X, Σ) .

Proof

Assume $A = \bigcup_{n=1}^{\infty} A_n$.

If $A = \bigcup_{k=1}^n B_k$ then

$$\sum_{k=1}^n |\nu(B_k)| = \sum_{k=1}^n \left| \sum_{m=1}^{\infty} \nu(B_k \cap A_m) \right| \leq \sum_{m=1}^{\infty} \sum_{k=1}^n |\nu(B_k \cap A_m)|$$

$$\leq \sum_{m=1}^{\infty} |v|(A_m).$$

To prove the opposite inequality it suffices to check that

$$\sum_{m=1}^M |v|(A_m) \leq |v|(A) \quad \forall M \in \mathbb{N}.$$

Since $|v|(A_m) \leq |v|(A)$, we may assume that $|v|(A_m) < \infty \quad \forall m$.
Fix M . For every $\varepsilon > 0$ we can find $n_m \in \mathbb{N}$ and $m \in M$ n_m such that

$$\text{and decomposition } A_m = \bigcup_{k=1}^{n_m} B_{m,k} \text{ s.t.}$$

$$\sum_{k=1}^{n_m} |v|(B_{m,k}) > |v|(A_m) - \varepsilon.$$

$$\text{Then } A = \left(\bigcup_{k=1}^M \bigcup_{k=1}^{n_m} B_{m,k} \right) \cup \left(A \setminus \bigcup_{m=1}^M A_m \right), \text{ so}$$

$$|v|(A) \geq \sum_{m=1}^M \sum_{k=1}^{n_m} |v|(B_{m,k}) + |v|(A \setminus \bigcup_{m=1}^M A_m)| \\ > \sum_{m=1}^M |v|(A_m) - \varepsilon M.$$

$$\text{hence } |v|(A) \geq \sum_{m=1}^{\infty} |v|(A_m).$$

It remains to check that $|v|(X) < \infty$.

Since $|v|(X) \leq |Re v|(X) + |Im v|(X)$,
we may assume that v is real-valued.

Assume $|v|(X) = \infty$.

we claim that if $A \in \Sigma$ is such that

$$|v|(A) = \infty, \text{ then } A = B \cup C \text{ s.t. } |v|(B) \geq 1$$

$$\text{and } |v|(C) = \infty. \text{ For this, find } A = \bigcup_{k=1}^{\infty} A_k \text{ s.t.}$$

$$\sum_{k=1}^{\infty} |v|(A_k) \geq 2 + |v|(A).$$

$$\text{Put } A' = \bigcup_{k: |v|(A_k) \geq 0} A_k, \quad A'' = \bigcup_{k: |v|(A_k) < 0} A_k. \text{ Then}$$

the above inequality means that

$$v(A') - v(A'') \geq 2 + |v(A') + v(A'')|.$$

This is possible only if

$$v(A') \geq 1, \quad v(A'') \leq -1.$$

As $|V|(A) = |V|(A') + |V|(A'')$, we have either $|V|(A') = \infty$ or $|V|(A'') = \infty$. In the first case we take $B = A''$, $C = A'$, otherwise take $B = A'$, $C = A''$. This proves the claim.

Using the claim we can now inductively choose mutually disjoint sets A_n s.t.

$$|V(A_n)| \geq 1 \text{ and } |V(X \setminus \bigcup_{k=1}^n A_k)| = \infty.$$

Then $\sum_{n=1}^{\infty} |V(A_n)|$ is divergent, on the other hand, it must be equal to $|V(\bigcup_{n=1}^{\infty} A_n)|$.

This is a contradiction. $\square$

Example

If $dV = f d\mu$ as in the previous example, then $d|V| = |f| d\mu$.

That $|V|(A) \leq \int_A |f| d\mu$ is immediate.

To prove the opposite inequality, fix $\varepsilon > 0$ and choose a simple function g s.t. $\|f - g\|_1 < \varepsilon$. Then g , assuming that $g = \sum_{k=1}^n \alpha_k \mathbb{1}_{A_k}$, we have

$$\begin{aligned} \int_A |f| d\mu &\leq \int_A |f - g| d\mu + \int_A |g| d\mu \\ &\leq \|f - g\|_1 + \sum_{k=1}^n |\alpha_k| \mu(A \cap A_k) \\ &\leq \varepsilon + \sum_{k=1}^n |\alpha_k| \mu(A \cap A_k). \end{aligned}$$

On the other hand,

$$\begin{aligned} \sum_{k=1}^n |\alpha_k| \mu(A \cap A_k) &= \sum_{k=1}^n \left| \int_{A \cap A_k} g d\mu \right| \\ &\leq \sum_{k=1}^n |V(A \cap A_k)| + \sum_{k=1}^n \left| \int_{A \cap A_k} (g - f) d\mu \right| \\ &\leq \sum_{k=1}^n |V(A \cap A_k)| + \|g - f\|_1. \end{aligned}$$

Therefore $\int_A |f| d\mu \leq 2\varepsilon + \sum_{k=1}^n |V(A \cap A_k)|$

better consider $g = \frac{f}{|f|}$
 $|V(A)| - |V(A')| \leq |V(A) - V(A')| = \|g - f\|_1$

A complex measure ν is said to be absolutely continuous w.r.t. a positive measure μ if $\mu(A)=0$ implies $\nu(A)=0$. We write $\nu \ll \mu$. By definition of $|\nu|$ we also get $|\nu| \ll \mu$.

Theorem (Radon-Nikodym theorem for complex measures)

Assume ν is a complex measure, μ a positive σ -finite measure, $\nu \ll \mu$. Then there exists a unique $f \in L^1(X, \Sigma, \mu)$ s.t. $d\nu = f d\mu$.

Proof. The uniqueness is clear. By considering $\operatorname{Re} \nu$ and $\operatorname{Im} \nu$ we may assume that ν is real-valued. Consider $|\nu|$.

Then $\nu_+ = \frac{|\nu| + \nu}{2}$ and $\nu_- = \frac{|\nu| - \nu}{2}$ are

positive measures, $\nu = \nu_+ - \nu_-$. Therefore

We may assume that ν is positive and then apply the usual Radon-Nikodym theorem. $\square$

Polar decomposition of complex measures

If ν is a complex measure, then there exists a unique positive measure μ (namely, $\mu = |\nu|$) and a measurable function $f: X \rightarrow \mathbb{T}$, uniquely defined up to a set of μ -measure zero, s.t. $d\nu = f d\mu$.

Proof

If $d\nu = f d\mu$ then $d|\nu| = |f| d\mu = d\mu$, so $\mu = |\nu|$. This proves uniqueness.

For the existence, apply the Radon-Nikodym theorem to $\nu \ll |\nu|$. We get a function $f \in L^1(X, \Sigma, |\nu|)$ s.t. $d\nu = f d|\nu|$.

It remains to check that $|f| = 1$ $|\nu|$ -a.e. This is clear, as then $d|\nu| = |f| d|\nu|$. $\square$