

Theorem (Hahn decomposition)

(3)

Assume ν is a (finite) signed measure on (X, Σ) .

Then there exists $N \in \Sigma$ s.t. the measures ν_- and ν_+ defined by

$$\nu_-(A) = -\nu(A \cap N), \quad \nu_+(A) = \nu(A \cap P) \quad (P = X \setminus N)$$

are finite and positive. The sets N and P are called the negative and, resp., positive sets of ν . They are unique modulo sets of $|\nu|$ -measure zero, and

$$\nu = \nu_+ - \nu_-, \quad |\nu| = \nu_+ + \nu_-.$$

The decomposition $X = N \cup P$ is called a Hahn decomposition.

Proof

Consider the polar decomposition

$$d\nu = f d|\nu|.$$

Since ν is real-valued, f is essentially real-valued. Hence modulo a set of $|\nu|$ -measure zero f takes two values: 1 and -1.

Then $N = \{x \mid f(x) = -1\}$ has the required properties.

If $X = \tilde{N} \cup \tilde{P}$ is another Hahn decomposition, then $\nu(N \cap \tilde{P}) \leq 0$ and $\nu(N \cap \tilde{P}) \geq 0$,

so $\nu(N \cap \tilde{P}) = 0$, and similarly $\nu(P \cap \tilde{N}) = 0$.

For the same reason $\nu(A) = 0$

for any $A \in \Sigma$ s.t. $A \subset P \cap \tilde{N}$ or $A \subset N \cap \tilde{P}$.

Thus $P \cap \tilde{N}$ and $N \cap \tilde{P}$ have $|\nu|$ -measure zero,

so $|\nu|(N \Delta \tilde{N}) = 0$.

Finally, $|\nu|(A) = |\nu|(A \cap N) + |\nu|(A \cap P)$
 $= -\nu(A \cap N) + \nu(A \cap P)$, so $|\nu| = \nu_+ + \nu_-$.

Equivalent form:

Theorem (Jordan decomposition)

Given a (finite) signed measure ν there exist unique mutually singular finite positive measures ν_+ and ν_- s.t.

$$\nu = \nu_+ - \nu_-$$

$$\text{(Namely, } \nu_+ = \frac{\nu + |\nu|}{2}, \nu_- = \frac{|\nu| - \nu}{2} \text{.)}$$

Remark

A more general notion of a signed measure allows values in $\mathbb{R} \cup \{+\infty\}$ or $\mathbb{R} \cup \{-\infty\}$.

For such measures one can still prove an analogue of the Hahn-Jordan decomposition.

In other words, ν is allowed to take values $+\infty$ or $-\infty$, but not both, so either $\nu: \mathcal{B} \rightarrow (-\infty, +\infty]$ or $\nu: \mathcal{B} \rightarrow \{-\infty, +\infty\}$.

Modes of convergence

Assume (X, Σ, μ) is a measure space,
 f, f_n are measurable functions on X .

We say that $\{f_n\}$ converges to f

in mean, if $\int_X |f - f_n| d\mu \xrightarrow{n \rightarrow \infty} 0$;

in measure, if $\mu(\{x \mid |f - f_n(x)| \geq \epsilon\}) \xrightarrow{n \rightarrow \infty} 0$
 $\forall \epsilon > 0$;

almost everywhere (a.e.), if $f_n(x) \xrightarrow{n \rightarrow \infty} f(x)$
for all x outside a set of measure zero.

Theorem

Assume f, f_n are measurable functions. Then:

(i) if $f_n \rightarrow f$ in mean, then $f_n \rightarrow f$ in measure;

(ii) if $f_n \rightarrow f$ a.e. and $\mu(X) < \infty$, then
 $f_n \rightarrow f$ in measure;

(iii) if $f_n \rightarrow f$ in measure ~~and $\mu(X) < \infty$~~ ,
then there exists a subsequence $\{f_{n_k}\}$
s.t. $f_{n_k} \rightarrow f$ a.e.

furthermore, $\forall \delta > 0$ there exists a subset $X_\delta \subset X$
of measure $< \delta$ s.t. $f_n \rightarrow f$ uniformly on $X \setminus X_\delta$
(Egorov's theorem)

Prove Egorov's theorem first

Proof

(i) The proof is based on the following inequality
(sometimes called Markov's inequality, or
Chebyshev's inequality):

If g is integrable and $\varepsilon > 0$, then

$$\mu(\underbrace{\{x \mid |g(x)| \geq \varepsilon\}}_A) \leq \frac{1}{\varepsilon} \int_A |g| d\mu.$$

Indeed, $|g| \geq \varepsilon \chi_A$, so $\int_A |g| d\mu \geq \varepsilon \mu(A)$.

Thus if $f_n \rightarrow f$ in mean then $\forall \varepsilon > 0$

$$\mu(\{x \mid |f(x) - f_n(x)| \geq \varepsilon\}) \leq \frac{1}{\varepsilon} \int |f - f_n| d\mu \xrightarrow{n \rightarrow \infty} 0.$$

(ii) For every $n, n \in \mathbb{N}$ put

$$X_{kn} = \bigcup_{j=n}^{\infty} \{x \mid |f(x) - f_j(x)| > \frac{1}{k}\}$$

For every k , $\{X_{kn}\}_n$ is a decreasing

sequence of sets and $\mu(\bigcap_n X_{kn}) = 0$

(as $\bigcap_n X_{kn}$ is contained in the set of x s.t. $f_j(x) \not\rightarrow f(x)$)

As $\mu(X_{k1}) \leq \mu(X) < \infty$, we conclude that

$$\mu(X_{kn}) \xrightarrow{n \rightarrow \infty} 0.$$

Fix $\varepsilon > 0$. For every k choose n_k s.t. $\mu(X_{kn_k}) < \frac{\varepsilon}{2^k}$

Put $X_\varepsilon = \bigcup_{k=1}^{\infty} X_{kn_k}$. Then

$$\mu(X_\varepsilon) < \sum_{k=1}^{\infty} \frac{\varepsilon}{2^k} = \varepsilon,$$

and for every k we have

$X \setminus X_\varepsilon \subset X \setminus X_{kn_k}$, so $|f(x) - f_j(x)| \leq \frac{1}{k}$

for all $x \in X \setminus X_\varepsilon$ and all $j \geq n_k$. Thus

$f_j \rightarrow f$ uniformly on $X \setminus X_\varepsilon$.

(iii) Assuming $f_n \rightarrow f$ in measure.

We want to show that a subsequence converges a.e.
Passing to a subsequence we may assume that

$$\mu(\underbrace{\{x \mid |f(x) - f_n(x)| \geq \frac{1}{2^n}\}}_A) < \frac{1}{2^n}.$$

Put $A = \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} A_n$. Then A has measure zero, (2)
and for every $x \notin A$ we have $x \notin A_n$ for
all n sufficiently large, so $|f(x) - f_n(x)| < \frac{1}{2^n}$
and therefore $f_n(x) \rightarrow f(x)$. $\square$