

Duals of L^p -spaces

Let $1 \leq p < \infty$ and $1 < q \leq \infty$ be s.t. $\frac{1}{p} + \frac{1}{q} = 1$.
We know that $(L^p)^* = L^q$. Our goal is to generalize this to measure spaces.

Let $(X, \mathcal{B}, \mu)$ be a ~~locally compact~~ measure space.

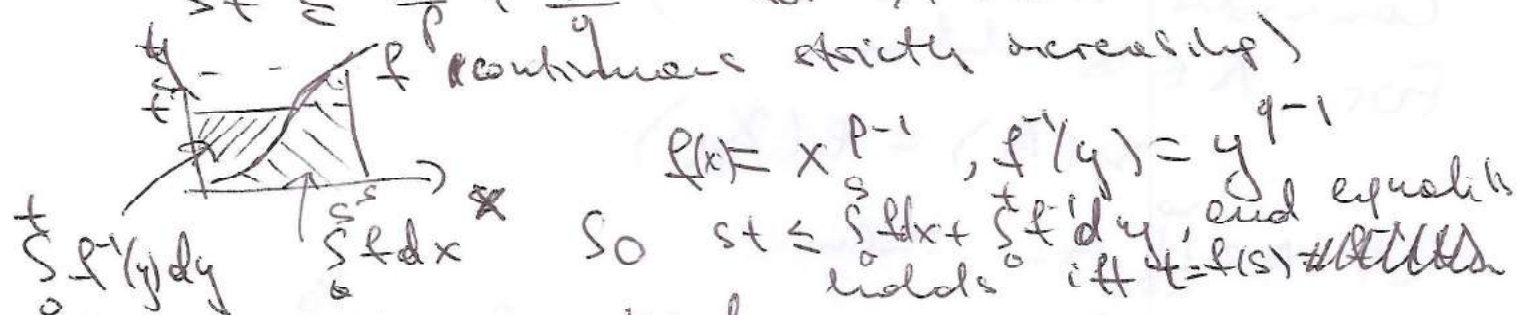
Recall that for $f \in L^p(X, \mu)$ and $g \in L^q(X, \mu)$ we have Hölder's inequality:

$$\|fg\|_1 \leq \|f\|_p \|g\|_q.$$

Proof?

We use Young's inequality:

$$st \leq \frac{s^p}{p} + \frac{t^q}{q} \quad \text{for } s, t \geq 0$$



It follows that

$$\frac{\|fg\|_1}{\|f\|_p \|g\|_q} \leq \frac{1}{p} \frac{\|f\|_p^p}{\|f\|_p^p} + \frac{1}{q} \frac{\|g\|_q^q}{\|g\|_q^q}$$

Integrating we get the result. Furthermore, we see that the equality holds if and only if for nonzero f, g and $(1 < p < \infty)$

$$\frac{\|f\|_p^p}{\|f\|_p^p} = \frac{\|g\|_q^q}{\|g\|_q^q} \quad \text{a.e., that is, } |g| = c |f|^{p-1}$$

This allows to define, for every $g \in L^q(X, \mu)$, a linear functional $\ell_g \in (L^p(X, \mu))^*$ by

$$e_g(f) = \int_X f g d\mu.$$

We then have $\|e_g\| = \|g\|_q$ if p (take $f = |g|^{q-1}$)
 and if $p=1$ and μ is σ -finite, then $\|e_g\| = \|g\|_\infty$, as we will see shortly.
 Therefore we get an isometric embedding
 map $L^q(X, d\mu)^* \hookrightarrow L^p(X, d\mu)^*$, $g \mapsto e_g$.

Theorem
 If $(X, \mathcal{B}, \mu)$ is a σ -finite measure space,
 $1 \leq p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$. Then the above
 map $L^q(X, d\mu)^* \hookrightarrow L^p(X, d\mu)^*$ is an isomorphism, so
 $L^p(X, d\mu)^* = L^q(X, d\mu)$.

Proof
 Consider first the case $\mu(X) < \infty$. Take $\psi \in L^p(X, d\mu)^*$.
 For $A \in \mathcal{B}$ define
 $\nu(A) = \psi(\chi_A)$.

If $A = \bigcup_{n=1}^{\infty} A_n$, then

$$\|\chi_A - \sum_{k=1}^n \chi_{A_k}\|_p = \mu(A \setminus \bigcup_{k=1}^n A_k)^{1/p} \xrightarrow{n \rightarrow \infty} 0.$$

hence ν is a complex measure on $(X, \mathcal{B})$.
 Clearly, $\nu \ll \mu$. Let $g = \frac{d\nu}{d\mu}$. Then

$$\psi(f) = \int_X f g d\mu$$

for any $f = \chi_A$, hence for any simple f .

Since simple functions are dense in
 $L^\infty(X, \mu) \subset L^p(X, d\mu)$ and $\|f\|_p \leq \|f\|_\infty \mu(X)^{1/p}$,
 it follows that

$$\psi(f) = \int_X f g d\mu \quad \forall f \in L^\infty(X, \mu) \subset L^p(X, d\mu).$$

We claim that this implies that $\|g\|_q \leq \|v\|$. Since $L^0(X, \mu)$ is dense in $L^p(X, d\mu)$ this then implies that $v = f g$.
Consider separately two cases:

$p \geq 1$

For $n \in \mathbb{N}$ put $g_n(x) = \begin{cases} g(x), & \text{if } |g(x)| \leq n \\ 0, & \text{otherwise} \end{cases}$

Then for $f_n = \frac{g_n}{\|g_n\|_q^{q-1}}$ we have

$$\|f_n\|_p = \|g_n\|_q^{\frac{q}{p}} = \|g_n\|_q^{q-1} \quad \text{and}$$

$$v(f_n) = \int_X |g_n|^q d\mu = \|g_n\|_q^q, \quad \text{so the inequality}$$

$$|v(f_n)| \leq \|v\| \cdot \|f_n\|_p$$

implies that $\|g_n\|_q \leq \|v\|$.

As $|g_n| \nearrow |g|$ μ -a.e., by the monotone

convergence theorem this implies that

$$\|g\|_q \leq \|v\|.$$

$p = 1$

For $c > 0$ consider the set $A = \{x \mid |g(x)| > c\}$.
If $\mu(A) > 0$, then for $f = \frac{\chi_A}{\mu(A)} \frac{g}{|g|}$ we

have

$$\|f\|_1 = 1, \quad v(f) = \frac{1}{\mu(A)} \int_A |g| d\mu > c.$$

But then the inequality

$$|v(f)| \leq \|v\| \cdot \|f\|_1$$

shows that we get a contradiction unless $c \leq \|v\|$.

Thus $\|g\|_\infty \leq \|v\|$.

Simply take $c = \|v\|$

This finishes the proof of theorem
for $\mu(X) < \infty$. For general σ -finite μ
choose an increasing sequence of
sets X_n of finite measure s.t. $X = \bigcup X_n$.
Then there exist $g_n \in L^p(X_n, d\mu)$ s.t.

$$\varphi = \lim_{n \rightarrow \infty} g_n \text{ a.s.} \quad \text{In particular,}$$

$$\|g_n\|_p \leq \|\varphi\|. \quad \text{We then have } g_{n+1} = g_n$$

μ -a.e. on X_n , so we can define g s.t.

$$g = g_n \quad \mu\text{-a.e. on } X_n \text{ for every } n.$$

$$\text{we have } \|g\|_p = \lim_{n \rightarrow \infty} \|g_n\|_p = \lim_{n \rightarrow \infty} \|\varphi\|_{L^p(X_n, d\mu)} = \|\varphi\|,$$

since $\bigcup L^p(X_n, d\mu)$ is dense in $L^p(X, d\mu)$.

For the same reason $\varphi = g$, since

$$\varphi(f) = g(f) \quad \forall f \in \bigcup L^p(X_n, d\mu). \quad \square$$

Warning

For $p = \infty$ we still get an isometric
embedding $L^p(X, d\mu) \hookrightarrow L^\infty(X, d\mu)^*$,
but generally it is far from being
surjective.