

Riesz representation theorem

Our goal is to describe $C_0(X)^*$ for locally compact X .

Def

A linear functional φ on $C_0(X)$ is called positive if $\varphi(f) \geq 0$ for $f \geq 0$.

(3)

Lemma

A positive linear functional on $C_0(X)$ is automatically bounded.

Proof

If φ is positive but unbounded,

we can find $f_n \geq 0$ s.t.

$$\|f_n\|_\infty < \frac{1}{2^n} \text{ but } \varphi(f_n) \geq 1.$$

Consider $f = \sum_{n=1}^{\infty} f_n$. Then $\varphi(f) \geq \sum_{n=1}^{\infty} \varphi(f_n)$

by positivity. This is a contradiction. $\square$

Def

A linear functional φ on $C_0(X)$ is called

real if $\varphi(\bar{f}) = \overline{\varphi(f)}$.

(Equivalently, $\varphi(f) \in \mathbb{R}$ if f is real-valued.)

For any $\varphi \in C_0(X)^*$ we can define

a linear functional $\bar{\varphi}$ by

$$\bar{\varphi}(f) = \overline{\varphi(\bar{f})}. \text{ Clearly, } \|\bar{\varphi}\| = \|\varphi\|.$$

Then the functionals $\varphi + \bar{\varphi}$ and $i(\varphi - \bar{\varphi})$ are real and

$$\varphi = \frac{\varphi + \bar{\varphi}}{2} + i \frac{\varphi - \bar{\varphi}}{2i}.$$

Thus the study of general linear functionals can be reduced to a study of real linear functionals. For real functionals, we have an analogue of the Jordan decomposition:

Theorem Proposition

Let X be a locally compact space and $\varphi \in C_0(X)^*$ be real. Then there exists positive $\varphi_+, \varphi_- \in C_0(X)^*$ s.t. $\varphi = \varphi_+ - \varphi_-$.

Proof

Define for $f \geq 0$

$$\varphi_+(f) = \sup \{ \varphi(g) \mid 0 \leq g \leq f \}$$

Note that $\varphi_+(f) \leq \|\varphi\| \|f\|$. We obviously have $\varphi_+(\lambda f) = \lambda \varphi_+(f)$ for $\lambda \geq 0$.

and $\varphi_+(f_1) + \varphi_+(f_2) \leq \varphi_+(f_1 + f_2)$. On

the other hand, if $0 \leq g \leq f_1 + f_2$,

put $g_1 = \min\{f_1, g\}$. Then $0 \leq g_1 \leq f_1$,

and for $g_2 = g - g_1 \geq 0$ we have

$$g_2 \leq \max\{g - f_1, 0\} \leq f_2$$

It follows that

$$\varphi(g) = \varphi(g_1) + \varphi(g_2) \leq \varphi_+(f_1) + \varphi_+(f_2).$$

Thus $\varphi_+(f_1 + f_2) = \varphi_+(f_1) + \varphi_+(f_2)$. Therefore

φ_+ uniquely extends to a positive linear functional on $C_0(X)$ (in the same way as the Lebesgue integral is first defined for positive functions and then extended to combinations of positive integrable functions). Then for $\varphi_- \stackrel{\text{def}}{=} \varphi_+ - \varphi \geq 0$ by definition of φ_+ .

Remark

It can be shown directly (and follows from the Hahn-Totter decomposition and Riesz's theorem below) that $\varphi_{\pm}$ defined in the proof satisfy $\|\varphi\| = \|\varphi_+\| + \|\varphi_-\|$, and the decomposition

Add exercise or extension on $V = (V_+ - V_-) \cap (V_+ - V_-)^\perp, (V_+ - V_-) \cap (V_+ - V_-)^\perp = 0$.

$\varphi = \varphi_+ - \varphi_-$ is uniquely characterized by this property. (1)

Any finite Borel measure μ on X defines a positive linear functional. We will show that this gives all positive linear functionals. We want to prove a more precise and stronger result. We need to slightly modify our notion of a Radon measure:

Def
A Borel measure μ on a locally compact space X is called Radon if it is outer regular, inner regular on open sets (so $\mu(U) = \sup_{K \subset U, \text{ compact}} \mu(K)$ for open U), and $\mu(K) < \infty$ for any compact $K \subset X$.

Proposition
If μ is a Radon measure on X then it is inner regular on every σ -finite Borel set.

In particular, if X is σ -compact ($X = \bigcup_{n=1}^{\infty} K_n$, $K_n \subset X$ compact), then μ is regular, so in this case there is no difference between our old and new defs.

Proof of proposition
It suffices to prove inner regularity on Borel sets of finite measure. Let A be such a set. Choose open U s.t.

$A \subset U$ and $\mu(U \setminus A) < \epsilon$. We
 can find a compact $F \subset U$ s.t. $\mu(U \setminus F) < \epsilon$.
 We can also find open V s.t.

$$U \setminus A \subset V \text{ and } \mu(V) < \epsilon.$$

Then $K = F \setminus V$ is compact,

$$K \subset U \setminus V \subset A \text{ and}$$

$$\mu(K) \leq \mu(A) \leq \mu(U) + \mu(V) < \mu(F) + \epsilon = \mu(K) + 2\epsilon,$$

$$\text{so } \mu(A \setminus K) < 2\epsilon.$$

Theorem (Riesz's, or Riesz-Markov's, theorem)

Let X be a locally compact space
 and φ be a positive linear functional
 on $C_c(X) \subset C_0(X)$. Then there exists
 a unique Radon measure μ on X s.t.

$$\varphi(f) = \int_X f d\mu \quad \text{if } f \in C_c(X).$$

Proof

For open $U \subset X$ define

$$\mu(U) = \sup \{ \varphi(f) ; 0 \leq f \leq \chi_U, \text{supp } f \subset U \}$$

and then for any $A \subset X$ define

$$\mu^*(A) = \inf_{\substack{A \subset U \\ U \text{ open}}} \mu(U).$$

Clearly, $\mu^*(U) = \mu(U)$ for open U .

We will show that μ^* is an outer measure
 on which then obtain the required measure
 by Carathéodory's theorem.

Step 1 μ^* is an outer measure.