

It suffices to show that (2)
 $\mu^*(\bigcup_{n=1}^{\infty} U_n) \leq \sum_{n=1}^{\infty} \mu(U_n)$ for open U_n .

(Indeed, if we are given sets $A_n \subset X$ with $\mu(A_n) < \infty \forall n$, then $\forall \epsilon > 0$ we can find open $U_n \supset A_n$ s.t. $\mu^0(A_n) \geq \mu(U_n) - \frac{\epsilon}{2^n}$, and then

$$\mu^*(\bigcup A_n) \leq \mu^*(\bigcup U_n) \leq \sum_n \mu^*(U_n) < \sum_n \mu^0(A_n) + \epsilon$$

Assume $f \in C_c(X)$, $0 \leq f \leq 1$, $\text{supp } f \subset \bigcup U_n$.

Put $K = \text{supp } f$. Consider ~~the~~ a partition of unity subordinate to the open cover $\{U_n\}_n$ ~~of K~~ of K .

Definition:

A (Hausdorff) topological space is called paracompact if every open cover admits a subordinate partition of unity. Every compact space is paracompact, as well as any second countable or metric space.

For open compact spaces this is easy:

If $U_1, \dots, U_n$ is a finite ^{open} cover of K , we first inductively choose open $V_1, \dots, V_n$

s.t. $\overline{V_i} \subset U_i$ but $V_1, \dots, V_n$ still covers K .

By Urysohn's lemma we then find $0 \leq f_k \leq 1$

s.t. $f_k = 1$ on $\overline{V_k}$ and $f_k = 0$ on U_k^c _{a neighbourhood of}

Then $\rho_k = \frac{f_k}{\sum_j f_j}$ form a partition of unity.

so we find n and $\rho_k \in C(K)$ s.t., $1 \leq k \leq n$,
s.t.

$0 \leq \rho_k \leq 1$, $\text{supp } \rho_k \subset U_k \cap K$, and $\sum_{k=1}^n \rho_k(x) = 1$
 $\forall x \in K$. Then put

$f_k = \rho_k f$ (so $f_k = 0$ if $x \notin K$).
 $\text{supp } f_k \subset U_k$ and $f = \sum_{k=1}^n f_k$, so

Then $0 \leq f_k \leq \chi_{U_k}$
 $\varphi(f) = \sum_{k=1}^n \varphi(f_k) \leq \sum_{k=1}^n \varphi(\chi_{U_k}) = \sum_{k=1}^n \mu(U_k)$.

hence

$$\mu\left(\bigcup_{k=1}^{\infty} U_k\right) \leq \sum_{k=1}^{\infty} \mu(U_k).$$

Step 2 Every open set is μ^* -measurable.
Let U be open. We have to show that if $A \subset X$, $\mu^*(A) < \infty$,

then $\mu^*(A) \geq \mu^*(A \cap U) + \mu^*(A \setminus U)$.

Let us first consider $A = V$ open. Fix $\varepsilon > 0$.

By definition there exists f s.t.

$0 \leq f \leq \chi_{V \cap U}$ and $\varphi(f) > \mu(V \cap U) - \varepsilon$.

Then $W = V \setminus \text{supp } f$ is open and compact.

$V \setminus U$, and we can find g s.t.

$0 \leq g \leq \chi_W$ and $\varphi(g) > \mu(W) - \varepsilon$.

Then $0 \leq f + g \leq \chi_{V \cap U} + \chi_W \leq \chi_V$, so
 $\text{supp } (f+g) \subset V$

$$\mu(V) \geq \varphi(f) + \varphi(g) \geq \mu(V \cap U) + \mu(W) - 2\varepsilon$$

$$\geq \mu(V \cap U) + \mu^*(V \setminus U) - 2\varepsilon.$$

Since this is true for any $\varepsilon > 0$, we
get the claim.

Now, for general A with $\mu^*(A) < \infty$ ③
 we can find $V \supset A$ open s.t. $\mu^*(V) < \mu^*(A) + \epsilon$.

Then

$$\begin{aligned} \mu^*(A) &> \mu(V) - \epsilon = \mu^*(V \cap U) + \mu^*(V \setminus U) \\ &\geq \mu^*(A \cap U) + \mu^*(A^c \setminus U). \end{aligned}$$

Again, as $\epsilon > 0$ was arbitrary, we get the claim.

It follows that all Borel sets are μ^* -measurable, so μ^* defines a Borel measure μ on X . By definition, μ is outer regular.

Step 3 The measure μ is inner regular on open sets, and for any compact $K \subset X$ we have

$$\mu(K) = \inf \{ \mu(U) \mid K \subset U, U \text{ open} \}$$

We start with the second statement.

~~For every open $U \subset X$ we can find $f \in C_c(X)$ s.t. $\text{supp } f \subset U$ and $0 \leq f \leq 1$.~~

Assume $K \subset U$. For $\epsilon > 0$ consider $U = \{x \mid f(x) > 1 - \epsilon\}$. Then U is open, $K \subset U$, and for any g with $0 \leq g \leq 1$ and $\text{supp } g \subset U$ we have $g \leq (1 - \epsilon)^{-1} f$, so $\int g = \frac{\int f}{1 - \epsilon}$. Hence $\mu(U) \leq \frac{\mu(K)}{1 - \epsilon}$.

and therefore

$$\mu^*(K) \leq \mu(U) \leq \frac{\mu(K)}{1 - \epsilon}.$$

As $\epsilon > 0$ was arbitrary, we get $\mu^*(K) \leq \mu(K)$.

On the other hand, for any $U \supset K$ open we can find f s.t. $0 \leq f \leq 1$ and $\text{supp } f \subset U$ (Urysohn's lemma again), and then

$$\mu^*(K) \leq \nu(f) \leq \mu(U).$$

As $\mu^*(K) = \inf \mu(U)$, we conclude that

$$\mu^*(K) = \inf \{ \nu(f) \mid 0 \leq f \leq 1, \text{supp } f \subset U \}.$$

Now, to prove outer regularity on open sets, fix such a set U and fix $c < \mu(U)$. Choose f s.t. $0 \leq f \leq 1$, and $\text{supp } f \subset U$ and $\nu(f) > c$. Let $K = \text{supp } f$. Then for any $g \geq f$ we have $g \geq f$, so $\nu(g) \geq \nu(f)$, and hence

$$\mu^*(K) \geq \nu(f) > c.$$

As $c < \mu(U)$ was arbitrary, this shows that

$$\mu(U) = \sup_{K \subset U} \mu^*(K).$$