

Step 4 $\varphi(f) = \int_X f d\mu$ for $f \in C_c(X)$.

It suffices to consider $0 \leq f \leq 1$.

Let $K = \text{supp } f$ and fix a number N .

Define $K_n = \{x \in K \mid f(x) \geq \frac{n}{N}\}$, $n = 0, \dots, N$.

Define $g_n = \min\{f, \frac{n}{N}\}$ and $(0 \leq n \leq N)$ (so $g_N = f$, $g_0 = 0$)

and $f_n = g_n - g_{n-1}$ ($1 \leq n \leq N$)

Then $f = \sum_{n=1}^N f_n$, ~~and~~ and

$$\frac{1}{N} \chi_{K_n} \leq f_n \leq \frac{1}{N} \chi_{K_{n-1}} \quad (1 \leq n \leq N). \quad (1)$$

It follows that

$$\frac{1}{N} \sum_{n=1}^N \mu(K_n) \leq \int_X f d\mu \leq \frac{1}{N} \sum_{n=0}^{N-1} \mu(K_n).$$

As $\frac{1}{N} \chi_{K_n} \leq f_n$, we have $\frac{1}{N} \mu(K_n) \leq \psi(f_n)$ by Step 1. On the other hand, if U is any open set containing K_{n-1} , then $0 \leq N f_n \leq 1$ and $\text{supp } f_n \subset U$ imply that $\psi(f_n) \leq \frac{1}{N} \mu(U)$, and hence $\mu \psi(f_n) \leq \frac{1}{N} \mu(K_{n-1})$ by outer regularity. Thus we also have

$$\frac{1}{N} \sum_{n=1}^N \mu(K_n) \leq \psi(f) \leq \frac{1}{N} \sum_{n=0}^{N-1} \mu(K_n)$$

It follows that

$$|\psi(f) - \int_X f d\mu| \leq \frac{\mu(K_0)}{N} = \frac{\mu(K)}{N}.$$

As N was arbitrary, we conclude that $\psi(f) = \int_X f d\mu$.

Step 5 Uniqueness of μ .

Assume μ is any ~~Radon~~ ~~Radon~~ measure c.t.

$$\int_X f d\mu = \psi(f) \quad \forall f \in C_c(X).$$

If U is open, then for any compact $K \subset U$ we can find f c.t. $\chi_K \leq f \leq \chi_U$. Then

$\mu(K) \leq \int \psi(f) \leq \mu(U)$. By inner regularity of μ on U , we conclude that $\mu(U) = \sup \{ \mu(K) \mid 0 \leq f \leq \chi_U \}$, so

$\mu(U)$ is uniquely determined by μ .
Then $\mu(A)$ is determined for any Borel A
by outer regularity. $\square$

Remark

Consider the smallest $\mathcal{B}$ -algebra $\mathcal{B}_0$ s.t. any
 $f \in C_c(X)$ is $\mathcal{B}_0$ -measurable. Equivalently,
 $\mathcal{B}_0$ is generated by the open sets
 $f^{-1}((0, +\infty))$ for $f \in C_c(X), f \geq 0$.

~~These sets are called $\mathcal{B}_0$ -sets~~
Elements of $\mathcal{B}_0$ are called Baire sets.

Then it can be shown that given
any Borel measure μ s.t. $\mu(K) < \infty$ for
compact K , the measure $\mu(A)$ of
any Baire set A contained in a union
of countably many compact sets is
determined by the integrals $\int_X f d\mu$.

If X is second countable, then
this gives all Borel sets A , so the
measure is automatically regular (this
we already know). But in general $\mathcal{B}_0$
does not coincide with the Borel $\mathcal{B}$ -algebra
and the measure μ is not determined
by integrals of $f \in C_c(X)$.

Example (Dieudonné)

Let X be the set of ordinals $\leq \omega_1$ (first
uncountable ordinal) with order topology:
the base of topology are the sets
 $\{x \mid \alpha < x < \beta\}$, for some $\alpha, \beta \in X$.
 $\{x \mid \alpha < x\}, \{x \mid x < \beta\}$

It can be shown that we can define a Borel measure μ on X by

$$\mu(A) = \begin{cases} 1, & \text{if } A \cup \{\omega_1\} \text{ contains an uncountable closed set} \\ 0, & \text{otherwise} \end{cases}$$

Then $\mu(\{\omega_1\}) = 0$, but $\mu(U) = 1$ for any open $U \ni \omega_1$. So μ is not ~~outer~~ regular.

The measure μ defines the functional $f \mapsto f(\omega_1)$ on $C(X)$. The Radon measure representing this functional is δ_{ω_1} .

We can now ~~compact~~ describe the dual of $C_0(X)$ in measure theoretic terms. Let us say that a complex ^{Borel} measure ν on X is ~~Rad~~ regular if the measures $(\operatorname{Re} \nu)_\pm$ and $(\operatorname{Im} \nu)_\pm$ are regular.

Consider the set $M(X)$ of regular complex Borel measures on X .

Proposition

$M(X)$ is a vector space with norm $\|\mu\| = |\mu|(X)$.

Furthermore, a complex Borel measure μ on X is regular if and only if $|\mu|$ is regular.

Proof (Remind that $|\mu \pm \nu| \leq |\mu| + |\nu|$ and $|\mu| \leq |\mu \pm \nu|$.)
Consider the vector space $M(X)$ consisting of linear combinations of finite regular measures.

We claim that if $\nu \in \tilde{M}(X)$, then $|\nu|$, $\nu_+(Re \nu)_+$ and $(Im \nu)_+$ are regular.

Note that ~~regularity of~~ a finite measure μ is regular if and only if $\forall$ Borel A and $\epsilon > 0$ there exist open U and compact K s.t.

$$K \subset A \subset U \text{ and } \mu(U \setminus K) < \epsilon$$

Since $(Re \nu)_+(U \setminus K), (Im \nu)_+(U \setminus K) \in |\nu|(U \setminus K)$, it follows that it suffices to show that $|\nu|$ is regular for $|\nu| \in \tilde{M}(X)$.

Assume $\nu = \sum_{n=1}^{\infty} \lambda_n \mu_n$ ($\lambda_n \in \mathbb{C}$).

Fix $\epsilon > 0$ and choose $K_n \subset A \subset U_n$ s.t. $\mu_n(U_n \setminus K_n) < \epsilon$. Then ~~for any decomposition~~ put $K = \bigcup_{n=1}^{\infty} K_n$ and $U = \bigcap_{n=1}^{\infty} U_n$.

We have $K \subset A \subset U$, and for any decomposition $U \setminus K = \bigcup_j B_j$, we get

$$\begin{aligned} \sum_j |\lambda_n \nu(B_j)| &\leq \sum_{j,n} |\lambda_n| \mu_n(B_j) \\ &= \sum_n |\lambda_n| \mu_n(U \setminus K) \\ &\leq \sum_n |\lambda_n| \mu_n(U_n \setminus K_n) \\ &\leq \epsilon \sum_n |\lambda_n|, \end{aligned}$$

so $|\nu|(U \setminus K) \leq \epsilon \sum_n |\lambda_n|$. Hence $|\nu|$ is regular.

So if we also let $\tilde{M}(X) = \{\nu \mid |\nu| \text{ is regular}\}$, then

$$M(X) \subset \tilde{M}(X) \subset \tilde{\tilde{M}}(X) \subset M(X)$$

$\uparrow$ def

$\uparrow$ just proved

$\uparrow$ since if $|\nu|$ is regular, then $(Re \nu)_+, (Im \nu)_+$ are regular.

Formulate these conclusions from the start?

Thus all three sets coincide. The claim that $|\nu|(X)$ is a norm is clear.

completing with $|\nu|(U \setminus K) \leq \sum_n |\lambda_n| \mu_n(U \setminus K)$

~~Exercise 13~~ For any $\mu \in M(X)$ we can define $\varphi_\mu \in C_0(X)^*$ by $\varphi_\mu(f) = \int_X f d\mu$.
Theorem

For any locally compact space X , the map $M(X) \rightarrow C_0(X)^*$, $\mu \mapsto \varphi_\mu$, is an isometric isomorphism, so $C_0(X)^* \simeq M(X)$.

Proof

Since $C_0(X)^*$ is spanned by positive linear functionals, to prove surjectivity it suffices to show that any positive φ has the form φ_μ for some μ .

By the Riesz-Markov theorem there exists a unique ~~Radon~~ Radon measure μ on X s.t. $\varphi(f) = \int_X f d\mu \quad \forall f \in C_c(X) \subset C_0(X)$.

By definition of μ we have

$$\mu(X) = \sup \{ \varphi(f) \mid 0 \leq f \leq 1, f \in C_c(X) \}$$

so μ is finite and hence regular (being Radon). Then φ_μ is well-defined, hence $\varphi = \varphi_\mu$.

Thus the map $M(X) \rightarrow C_0(X)^*$ is surjective, and it remains only to check

$$\|\varphi_\nu\| = \|\nu\|(X) \quad \forall \nu \in M(X).$$

~~It is clear that the inequality~~ The inequality

$$\|\varphi_\nu\| \leq \|\nu\|(X) \text{ is clear, since}$$

$$\left| \int_X f d\nu \right| \leq \int_X |f| d|\nu| \leq \|f\| \|\nu\|(X).$$

The opposite inequality is also clear for positive ν , as $\|\nu\|(X) = \nu(X) = \sup \{ \varphi_\nu(f) \mid 0 \leq f \leq 1 \}$.

For general ν consider the polar

$$\text{decomposition } d\nu = g d\mu$$

(so $\mu = |\nu|$, $g: X \rightarrow \mathbb{C}$).

By regularity of μ the space $C_c(X)$ is dense in $L^1(X, d\mu)$

(We already had a similar statement for metric space. here the argument is the same similar).

For open U , ~~similar~~ by the proof of the Riesz-Mazur theorem we have

$$\mu(U) = \sup \left\{ \int_U f d\mu \mid 0 \leq f \in X_U \right\}$$

hence the closure of $C_c(X)$ in $L^1(X, d\mu)$

contains X_U for all open U ~~with $\mu(U) > 0$~~ .

Then by outer regularity of μ it contains

all X_A for Borel A . This, in turn,

implies that it coincides with $L^1(X, d\mu)$.

Since a convergent sequence in $L^1(X, d\mu)$ contains a μ -a.e. convergent subsequence,

it follows that we can choose

$f_n \in C_c(X)$ s.t. $f_n \rightarrow \bar{g}$ μ -a.e.

Define $g_n \in C_c(X)$ by

$$g_n(x) = \begin{cases} f_n(x), & \text{if } |f_n(x)| \leq 1 \\ \frac{f_n(x)}{|f_n(x)|}, & \text{otherwise.} \end{cases}$$

Then $|g_n| \leq 1$ and $g_n \rightarrow \bar{g}$ μ -a.e.,

hence $g_n \rightarrow g$ in $L^1(X, d\mu)$ by the dominated convergence theorem. Then

$$\varphi(g_n) = \int_X g_n g d\mu \xrightarrow{n \rightarrow \infty} \int_X 1 d\mu = \mu(X) = \|1\|_1,$$

so $\|1\|_1(X) \leq \|\varphi\|$ as $\|g_n\| \leq 1$. □

Remark For $X=(0,1)$ and $dV = f d\lambda$, $f \in L^1(0,1)$, the claim that $\|\varphi_V\| = \|1\|_1(X) = \int_0^1 |f| d\lambda = \|f\|_1$ was used in one of the first lectures, when we discussed Fourier series.