

15.11.21

(1)

Consider $g \in C^\infty(\mathbb{R})$ s.t. $g(x) = 0$ for $x \leq 0$,
 $g(x) > 0$ for $x > 0$. Then for

$$f(x) = \frac{g(x)}{g(x) + g(1-x)}$$
 we have

$f \in C^\infty(\mathbb{R})$, $0 \leq f \leq 1$, $f(x) = 0$ for $x \leq 0$,
 $f(x) = 1$ for $x \geq 1$

Recall that we proved that

$$\widehat{\partial^\alpha f}(y) = (2\pi i y)^\alpha \widehat{f}(y).$$

Similarly, we have

Proposition

Denote by w_α the function $w_\alpha(x) = x^\alpha$.
 Assume $(1+|x|)^N f(x) \in L^1(\mathbb{R}^n)$ for
 some $N \geq 1$. Then $\widehat{f} \in C^N(\mathbb{R}^n)$ and

$$(-2\pi i)^{|\alpha|} w_\alpha \widehat{f}(y) = (\partial^\alpha \widehat{f})(y) \text{ for } |\alpha| \leq N.$$

Proof

$$\begin{aligned} \frac{\partial \widehat{f}}{\partial y_j}(y) &= \lim_{t \rightarrow 0} \frac{\widehat{f}(y + te_j) - \widehat{f}(y)}{t} \\ &= \lim_{t \rightarrow 0} \int_{\mathbb{R}^n} f(x) \frac{e^{-2\pi i(y+te_j) \cdot x} - e^{-2\pi i y \cdot x}}{t} dx \\ &= \lim_{t \rightarrow 0} \int_{\mathbb{R}^n} f(x) e^{-2\pi i y \cdot x} \frac{e^{-2\pi i t x_j} - 1}{t} dx \quad (*) \end{aligned}$$

We have $|e^{-2\pi i t x_j} - 1| \leq 2\pi |t x_j|$
 Therefore the integrand is dominated by
 $2\pi |x_j| f(x) \in L^1(\mathbb{R}^n)$. Hence, by the

dominated convergence theorem, we have

$$\ominus -i \int_{\mathbb{R}^n} f(x) x_j e^{-i y \cdot x} dx.$$

This proves the proposition for
 $\delta = (0, \dots, 0, 1, 0, \dots, 0)$.

The general case is by induction on $|\delta|$. 19

Corollary

If $f \in S(\mathbb{R}^n)$, then $\hat{f} \in S(\mathbb{R}^n)$.

Recall that

$$(f * g)(x) = \int_{\mathbb{R}^n} f(x-y) g(y) dy = \int_{\mathbb{R}^n} g(x-y) f(y) dy.$$

$$\|f * g\|_1 \leq \|f\|_1 \|g\|_1.$$

However, $f * g$ makes sense for other pairs, for example,

$$f \in C_c(\mathbb{R}^n), \quad g \in L^1_{loc}(\mathbb{R}^n)$$

In the assignment it was shown that if $\phi \in C_c(\mathbb{R}^n)$ and $f \in L^1_{loc}(\mathbb{R}^n)$, then

$$\phi * f \in C(\mathbb{R}^n)$$

Similarly, it is not difficult to check that if $\phi \in C_c^\infty(\mathbb{R}^n)$ and $f \in L^1_{loc}(\mathbb{R}^n)$,

then $\phi * f \in C^\infty(\mathbb{R}^n)$ and

$$\partial^\alpha (\phi * f) = (\partial^\alpha \phi) * f.$$

The following was almost shown in the assignment:

Proposition

②

Assume $(\phi_n)_{n=1}^\infty$ is a sequence in $L^1(\mathbb{R}^n)$ s.t.

$$\phi_n \geq 0, \int_{\mathbb{R}^n} \phi_n dx = 1, \int_{\mathbb{R}^n \setminus B_r(0)} \phi_n dx \xrightarrow{n \rightarrow \infty} 0$$

for every $r > 0$. Then, for every $f \in L^1(\mathbb{R}^n)$,
 $\phi_n * f \rightarrow f$ in $L^1(\mathbb{R}^n)$ ~~as $n \rightarrow \infty$~~ .

Proof

Put $f_x(y) = f(x+y)$. From the assignment we know that $x \mapsto f_x \in L^1(\mathbb{R}^n)$ is continuous. Fix $\varepsilon > 0$ and take $\delta > 0$ s.t.

$$\|f - f_x\|_1 < \varepsilon \text{ if } |x| < \delta. \text{ ~~Put } x = y \text{ in } B_\delta(0)~~$$

Then

$$\begin{aligned} \|\phi_n * f - f\|_1 &= \int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} \phi_n(y) f(x-y) dy - f(x) \right| dx \\ &= \int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} \phi_n(y) (f(x-y) - f(x)) dy \right| dx \\ &\leq \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} \phi_n(y) |f(x-y) - f(x)| dx \right) dy \\ &= \int_{\mathbb{R}^n} \phi_n(y) \|f_y - f\|_1 dy \\ &= \int_{B_\delta(0)} \phi_n(y) \|f_y - f\|_1 dy + \int_{\mathbb{R}^n \setminus B_\delta(0)} \phi_n(y) dy \\ &< \varepsilon \int_{B_\delta(0)} \phi_n(y) dy + 2\|f\|_1 \int_{\mathbb{R}^n \setminus B_\delta(0)} \phi_n(y) dy \\ &\quad \downarrow \quad \downarrow \\ &< \varepsilon \quad 0 \\ &< \varepsilon \end{aligned}$$

for n large enough

□

In particular, taking $\varphi_k \in C_c^\infty(\mathbb{R}^n)$,
we can approximate $f \in L^1(\mathbb{R}^n)$ by
 $\varphi_k * f \in L^1(\mathbb{R}^n) \cap C^\infty(\mathbb{R}^n)$.

Often one takes $\varphi \in C_c^\infty(\mathbb{R}^n)$ s.t.

$$\varphi \geq 0, \quad \int_{\mathbb{R}^n} \varphi(x) dx = 1,$$

and then put

$$\varphi_k(x) = \frac{1}{k^n} \varphi\left(\frac{x}{k}\right)$$

Such φ is called a positive mollifier.

Returning to the Fourier transform,
an important property is

$$\widehat{f \otimes g} = \hat{f} \cdot \hat{g},$$

which follows from the Fubini theorem.

Fourier inversion theorem

Theorem $f \in L^1(\mathbb{R}^n)$ is s.t. $\hat{f} \in L^1(\mathbb{R}^n)$.

Then $f(x) = \int_{\mathbb{R}^n} \hat{f}(y) e^{2\pi i x \cdot y} dy$ for a.e. x .

The theorem applies to $f \in C_c^\infty(\mathbb{R}^n)$,
more generally, to $f \in S(\mathbb{R}^n)$, but, as we
will see later, to many more functions.

Note that $C_c^\infty(\mathbb{R}^n)$ is dense in $L^1(\mathbb{R}^n)$, since

$C_c(\mathbb{R}^n)$ is dense in $L^1(\mathbb{R}^n)$, and if $f \in C_c(\mathbb{R}^n)$,
then $\varphi * f \in C_c^\infty(\mathbb{R}^n)$ for any $\varphi \in C_c^\infty(\mathbb{R}^n)$.

Key Lemma

For any $f, g \in L^1(\mathbb{R}^n)$ we have

$$\int_{\mathbb{R}^n} \widehat{f} \cdot g \, dx = \int_{\mathbb{R}^n} f \cdot \widehat{g} \, dx$$

Proof

By the Fubini theorem both sides are equal to

$$\int_{\mathbb{R}^n \times \mathbb{R}^n} f(x) g(y) e^{-2\pi i x \cdot y} \, dx \, dy.$$

Proof of the theorem

Take any function $\phi \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$

Set. (i) ϕ is continuous at 0, $\phi(0) = 1$;

$$(ii) \widehat{\phi} \geq 0;$$

$$(iii) c \stackrel{\text{def}}{=} \int_{\mathbb{R}^n} \widehat{\phi}(y) \, dy < \infty.$$

For example, $\phi(x) = e^{-\pi|x|^2}$, then $\widehat{\phi} = \phi$

and $c = 1$.

For a fixed $x \in \mathbb{R}^n$, consider the function

$g_{x,x}$ defined by

$$g_{x,x}(y) = \phi\left(\frac{y}{K}\right) e^{2\pi i x \cdot y}.$$

Apply our key lemma to f and $g_{x,x}$:

$$\int_{\mathbb{R}^n} \widehat{f}(y) \phi\left(\frac{y}{K}\right) e^{2\pi i x \cdot y} \, dy$$

$$= \int_{\mathbb{R}^n} f(y) \widehat{g_{x,x}}(y) \, dy. \quad (*)$$

Let $K \rightarrow \infty$.

As ϕ is bounded and $\phi\left(\frac{y}{K}\right) \rightarrow \phi(0) = 1 \, \forall y$,
the left hand side converges to $\int_{\mathbb{R}^n} \widehat{f}(y) e^{2\pi i x \cdot y} \, dy$.

Therefore we only need to show that the right hand side converges to $f(x)$ for a.e. x .

Let us let

$$(\hat{\varphi})_n(y) = n^n \hat{\varphi}(ny), \text{ then}$$

$$\hat{g}_{n,x}(y) = (\hat{\varphi})_n(x-y).$$

Therefore the right hand side $n(x)$ is

$$\int_{\mathbb{R}^n} f(y) (\hat{\varphi})_n(x-y) dy = ((\hat{\varphi})_n * f)(x).$$

Since $\int_{\mathbb{R}^n} (\hat{\varphi})_n(y) dy = \int_{\mathbb{R}^n} \hat{\varphi}(y) dy = c$ and

$$\int_{\mathbb{R}^n \setminus B_{n^k}(0)} (\hat{\varphi})_n(y) dy = \int_{\mathbb{R}^n \setminus B_{n^k}(0)} \hat{\varphi}(y) dy \xrightarrow{n \rightarrow \infty} 0,$$

we have $(\hat{\varphi})_n * f \rightarrow cf$ in $L^1(\mathbb{R}^n)$.

Passing to a subsequence, we have

$$(\hat{\varphi})_{n_m} * f \rightarrow cf \text{ a.e. hence}$$

$$\int_{\mathbb{R}^n} \hat{f}(y) e^{i\tilde{u}x \cdot y} dy = cf(x)$$

for a.e. x . Since $c=1$ for $\varphi(x) = e^{-\tilde{u}|x|^2}$,
we get the result. $\square$