

UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

New/Deferred examination in: MAT3400/4400 — Linear analysis with applications

Day of examination: Friday, January 15, 2016.

Examination hours: 14.30 – 18.30.

This problem set consists of 3 pages.

Appendices: None.

Permitted aids: None.

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Note: You must justify all your answers!

Problem 1

(15 points) Let $(\Omega, \mathcal{A}, \mu)$ be a measure space and $\{f_n\}_{n \in \mathbb{N}}$ a sequence of $\mathcal{A}$ -measurable functions such that f_n converges μ -a.e. to a measurable function f . Suppose that there is an integrable nonnegative function g defined on Ω such that $|f_n| \leq g$ for all $n \in \mathbb{N}$. Show that

$$\lim_{n \rightarrow \infty} \int_{\Omega} |f_n - f| d\mu = 0.$$

Problem 2

(15 points) Consider the measure space $X = (0, 1]$ with the Borel σ -algebra $\mathcal{B}$ on $(0, 1]$ and the Lebesgue measure λ . Form the product measure $\lambda \otimes \lambda$ on the product σ -algebra $\mathcal{B} \otimes \mathcal{B}$. It is assumed known that $\mathcal{B} \otimes \mathcal{B}$ is equal to the Borel σ -algebra of subsets of $X \times X$. Let $f : X \times X \rightarrow \mathbb{R}$ be the continuous function given by $f(x, y) = \frac{1}{x+y}$. Prove that f is $\mathcal{B} \otimes \mathcal{B}$ -measurable and that

$$\int_{X \times X} f(x, y) d(\lambda \otimes \lambda)(x, y) = 2 \ln(2).$$

(Hint: For each $n \geq 1$ consider the restriction of f to the measurable set $[\frac{1}{n}, 1] \times [\frac{1}{n}, 1]$. Use the fact that for a continuous function u we have $\int \ln(u) du = u \ln(u) - u$ for the indeterminate Riemann integral.)

Problem 3

Let $l^2(\mathbb{N})$ denote the Hilbert space of complex-valued sequences $\{x_n\}_{n \in \mathbb{N}}$ such that $\sum_{n=1}^{\infty} |x_n|^2 < \infty$. Let $\{c_n\}_{n \in \mathbb{N}}$ be a bounded sequence of complex

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numbers. Define $T : l^2(\mathbb{N}) \rightarrow l^2(\mathbb{N})$ by $T(\{x_n\}_{n \in \mathbb{N}}) = \{c_n x_n\}_{n \in \mathbb{N}}$.

3a

(15 points) Show that T is bounded and find the adjoint operator T^* . Conclude that T is a normal operator.

3b

(10 points) Show that for each $n \geq 1$, the complex number c_n is an eigenvalue of T .

Problem 4

4a

(15 points) Suppose that H is an infinite dimensional Hilbert space with an orthonormal basis $\{e_n\}_{n \in \mathbb{N}}$. Suppose that $T \in B(H)$ such that $\sum_{n=1}^{\infty} \|T(e_n)\|^2 < \infty$. For each $m \geq 1$ define a linear operator by

$$T_m(f) = \sum_{j=1}^m (f | e_j) T(e_j)$$

for every $f \in H$. Prove that for each $m \geq 1$,

$$\|(T_m - T)(f)\| \leq \|f\| \left(\sum_{n=m+1}^{\infty} \|T(e_n)\|^2 \right)^{1/2},$$

and use this to show that $\lim_{m \rightarrow \infty} \|T_m - T\| = 0$.

Let $(X, \mathcal{A}, \mu)$ be a finite measure space. Denote by H the Hilbert space $L^2(X, \mu)$. Recall that the inner-product is given by

$$(g, h) = \int_X g(y) \overline{h(y)} d\mu(y)$$

for $g, h \in H$. Form the product measure $\mu \otimes \mu$ on the σ -algebra $\mathcal{A} \otimes \mathcal{A}$ of subsets of $X \times X$. Suppose that $k : X \times X \rightarrow \mathbb{C}$ is a bounded $\mathcal{A} \otimes \mathcal{A}$ -measurable function. Let k_x be the section of k defined as $k_x(y) = k(x, y)$ for (x, y) in $X \times X$.

4b

(10 points) Explain why $k \in L^2(X \times X, \mu \otimes \mu)$. Conclude that $k_x \in H$ for μ -a.e. $x \in X$. (Use Fubini's theorem.)

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4c

(10 points) Consider the bounded operator $T \in B(H)$ given by

$$(Tf)(x) = \int_X k(x, y) f(y) d\mu(y)$$

for $f \in H$. Suppose that $\{f_n\}_{n \in \mathbb{N}}$ is an orthonormal basis for H , and let $e_n(y) = \overline{f_n(y)}$ for all $n \geq 1$, $y \in X$ (use without proof that $\{e_n\}$ is again an orthonormal basis). Prove that

$$\int_X \int_X |k(x, y)|^2 d\mu(x) d\mu(y) = \sum_{n=1}^{\infty} \|T(e_n)\|^2.$$

(Use problem 4b and Parseval's identity for the sequence $\{f_n\}_{n \in \mathbb{N}}$ in H .)

4d

(10 points) Conclude that the operator T defined in problem 4c is a compact operator.

END