

UNIVERSITY OF OSLO

Faculty of Mathematics and Natural Sciences

Examination in MAT4410 — Advanced linear analysis.

Day of examination: Monday, December 17, 2012.

Examination hours: 9.00–13.00.

This problem set consists of 3 pages.

Appendices: None.

Permitted aids: None.

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Note: You must justify all your answers!

Problem 1

(15 points) Let λ denote Lebesgue measure on $X = [0, \infty)$. Find the limit

$$\lim_{n \rightarrow \infty} \int_{X \times X} e^{-(\frac{x^2 y^2}{n} + x + y)} d(\lambda \otimes \lambda)(x, y).$$

Justify your answer.

Problem 2

Let c_0 denote the Banach space of sequences of complex numbers converging to zero. Let $a = \{a_n\}_{n \geq 1}$ be a sequence of complex numbers such that $\sum_{n=1}^{\infty} a_n b_n$ is convergent for every $\{b_n\}_{n \geq 1} \in c_0$. For every $k \geq 1$, define $T_k : c_0 \rightarrow \mathbb{C}$ by $T_k(b) = \sum_{j=1}^k a_j b_j$ for $b = \{b_n\}_{n \geq 1}$ in c_0 .

2a

(10 points) Show that T_k is a bounded linear functional for every $k \geq 1$.

2b

(10 points) Conclude that $a \in l^1(\mathbb{N})$. What is the relationship between $\|a\|_1$ and $\|T_k\|$ for $k \geq 1$?

(Continued on page 2.)

Problem 3

3a

(5 points) Formulate a consequence of the Hahn-Banach extension theorem for linear functionals appropriate for linear subspaces of normed spaces.

3b

(10 points) If X is a normed space and $Y \subset X$ is a linear subspace, show that Y is dense in X if and only if the only element $\varphi \in X^*$ such that $\varphi(y) = 0$ for all $y \in Y$ is the zero functional $\varphi_0(x) = 0$ for all $x \in X$.

3c

(10 points) Suppose that μ is a Borel measure on $[0, 1]$ such that $\int_{[0,1]} x^k d\mu(x) = 0$ for all $k \geq 1$. Show that $\mu = 0$.

3d

(10 points) (This problem is independent of problems 3b and 3c.) Let λ be the Lebesgue measure on $[0, 1]$ and let $L^p([0, 1], \lambda)$ for $1 < p < \infty$ be the Banach space of p -integrable, complex-valued functions on $[0, 1]$. Suppose that $\{f_k\}_{k \geq 1}$ is a sequence of elements in $L^p([0, 1], \lambda)$ and $\{a_k\}_{k \geq 1}$ is a sequence of complex numbers for which there exists $C > 0$ such that

$$\left| \sum_{k=1}^m \alpha_k a_k \right| \leq C \left(\int_{[0,1]} \left| \sum_{k=1}^m \alpha_k f_k(x) \right|^p d\lambda(x) \right)^{1/p}$$

for any choice of complex numbers $\alpha_1, \dots, \alpha_m$ and any $m \geq 1$. Let $\frac{1}{p} + \frac{1}{q} = 1$. Show that there is $g \in L^q([0, 1], \lambda)$ such that

$$\int_{[0,1]} f_k(x) g(x) d\lambda(x) = a_k, \quad \forall k \geq 1.$$

Problem 4

4a

(10 points) Let (X, Σ) be a measurable space and λ, μ be measures on Σ such that λ is finite, μ is σ -finite and $\lambda \ll \mu$. Prove that for every $\varepsilon > 0$ there is $\delta > 0$ such that

$$\mu(A) < \delta \Rightarrow \lambda(A) < \varepsilon, \quad \forall A \in \Sigma.$$

(Continued on page 3.)

4b

(10 points) Let $X = \mathbb{R}$ with its Borel σ -algebra $\mathcal{B}$ and let λ be Lebesgue measure on $\mathcal{B}$. Define a measure on $\mathcal{B}$ by

$$\mu(A) = \int_A \frac{1}{e^{|x|}} d\lambda(x) \text{ for all } A \in \mathcal{B}.$$

Explain why $\lambda \ll \mu$.

4c

(10 points) With $(\mathbb{R}, \mathcal{B})$, λ and μ as in question 4b, show that the conclusion of problem 4a does not hold.

END