

UNIVERSITY OF OSLO

Faculty of Mathematics and Natural Sciences

Examination in: MAT4410 — Advanced linear analysis

Day of examination: 01.12.2014

Examination hours: 09.00–13.00

This problem set consists of 3 pages.

Appendices: None

Permitted aids: None

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Problem 1

1a

Let X be a normed space and $Y \subset X$ be a subspace. Show that the closure of Y in X coincides with

$$\bigcap_{f \in X^*: Y \subset \ker f} \ker f.$$

1b

Consider the space ℓ^p for some $1 < p < \infty$ and the subspace $V \subset \ell^p$ consisting of all sequences $x = (x_n)_n \in \ell^p$ such that $x_n \neq 0$ only for finitely many n 's and $\sum_n x_n = 0$. Show that V is dense in ℓ^p .

Problem 2

Let X be a locally compact space (not necessarily second countable) and μ be a Radon measure on it.

2a

Show that there exists the largest open set $U \subset X$ such that $\mu(U) = 0$. Its complement $X \setminus U$ is called the *support* of μ and denoted by $\text{supp } \mu$.

2b

Show that the set $U = X \setminus \text{supp } \mu$ can be characterized as the largest open set such that if $f \in C_c(X)$ and $\text{supp } f \subset U$, then $\int_X f d\mu = 0$.

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Problem 3

Let $(X, \mathcal{B}, \mu)$ be a σ -finite measure space.

3a

Let ν be another σ -finite measure on $(X, \mathcal{B})$ such that $\nu \ll \mu$. Show that for any measurable function $f: X \rightarrow [0, +\infty]$ we have

$$\int_X f d\nu = \int_X f \frac{d\nu}{d\mu} d\mu.$$

3b

Show that for $1 \leq p < \infty$ and a measurable function $f: X \rightarrow \mathbb{C}$ we have $f \in L^p(X, d\nu)$ if and only if $(d\nu/d\mu)^{1/p} f \in L^p(X, d\mu)$. What can you suggest as a replacement of this statement for $p = \infty$?

3c

Show that if ν is as above and η is another σ -finite measure on $(X, \mathcal{B})$ such that $\eta \ll \nu$, then

$$\frac{d\eta}{d\mu} = \frac{d\eta}{d\nu} \cdot \frac{d\nu}{d\mu} \quad \mu\text{-a.e.}$$

3d

Let $T: X \rightarrow X$ be a measurable map such that the measures $T_*\mu$ and μ are equivalent, so $\mu(T^{-1}(A)) = 0$ if and only if $\mu(A) = 0$. For $1 \leq p < \infty$ define an operator U_T on $L^p(X, d\mu)$ by

$$(U_T f)(x) = \frac{d\mu}{dT_*\mu}(T(x))^{1/p} f(T(x)).$$

Show that U_T is an isometry, that is, $\|U_T f\|_p = \|f\|_p$ for all f .

Problem 4

Let $(X, \mathcal{B}, \mu)$ be a measure space, $\mu(X) < \infty$.

4a

Assume $V \subset L^\infty(X, \mu)$ is a subspace such that $\|f\|_\infty \leq C\|f\|_2$ for all $f \in V$ for some $C > 0$. Show that

$$\dim V \leq C^2 \mu(X).$$

Hint: take an orthonormal system $\{f_k\}_{k=1}^n$ in V and try to get an estimate on n .

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4b

Show that if $V \subset L^2(X, d\mu)$ is an infinite-dimensional closed subspace, then $V \not\subset L^\infty(X, \mu)$.

THE END