

UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Exam in: MAT4410 — Advanced linear analysis

Day of examination: December 10, 2024

Examination hours: 15.00–19.00

This problem set consists of 3 pages.

Appendices: None.

Permitted aids: None.

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

The points in parentheses indicate the maximum possible score for each problem or subproblem. If you are unable to solve a subproblem, you may assume the result of that problem when solving later problems. E.g., if you cannot solve problem 2a, you may assume the result of 2a and try to solve 2b.

Note: You must justify all your answers!

Problem 1 (weight 40 points)

Let H be a Hilbert space (over $\mathbb{R}$ or $\mathbb{C}$) and $P \in \mathcal{B}(H)$. Assume P satisfies that $P^2 = P$, and let M denote the range of P , i.e., $M = P(H)$. Note that we don't assume that P is self-adjoint.

1a (weight 10 points)

Show that M is closed in H .

1b (weight 10 points)

Show that if P is compact, then P has finite-rank.

1c (weight 10 points)

Assume there exists $U \in \mathcal{B}(H)$ such that $U^*U = P$.

Show that P is self-adjoint and U is isometric on M (that is, $\|U(y)\| = \|y\|$ for all $y \in M$). Show also that U annihilates $M^\perp$ (that is, $M^\perp \subseteq \ker(U)$).

(Continued on page 2.)

1d (weight 10 points)

Assume $V \in \mathcal{B}(H)$ is isometric on a closed subspace N of H , and that V annihilates $N^\perp$. Let Q denote the orthogonal projection of H onto N . Show that

$$\langle (V^*V)(x), x \rangle = \langle Q(x), x \rangle \quad \text{for all } x \in H.$$

Then deduce that $V^*V = Q$.

Problem 2 (weight 30 points)

We recall that $H := \ell^2(\mathbb{N})$ is a complex Hilbert space, with inner product given by

$$\langle \xi, \eta \rangle = \sum_{n=1}^{\infty} \xi(n) \overline{\eta(n)}, \quad \xi, \eta \in H.$$

We let $\{\delta_k\}_{k \in \mathbb{N}}$ denote the standard orthonormal basis of H , that is, for each $k \in \mathbb{N}$, $\delta_k \in H$ is given by

$$\delta_k(n) = \begin{cases} 1 & \text{if } n = k, \\ 0 & \text{if } n \neq k \end{cases}$$

for $n \in \mathbb{N}$. In this exercise we consider the bounded linear operator $S : H \rightarrow H$ satisfying

$$[S(\xi)](n) = \frac{1}{n} \xi(n+1) \quad \text{for all } \xi \in H \text{ and } n \in \mathbb{N}.$$

2a (weight 10 points)

Check that $S(\delta_1) = 0$ and $S(\delta_k) = \frac{1}{k-1} \delta_{k-1}$ for all $k \in \mathbb{N}, k \geq 2$. Then show that $S^*(\delta_k) = \frac{1}{k} \delta_{k+1}$ for all $k \in \mathbb{N}$.

2b (weight 10 points)

Show that S is a Hilbert-Schmidt operator. Then find a sequence $\{\sigma_j\}_{j \in \mathbb{N}}$ of positive real numbers and orthonormal sequences $\{u_j\}_{j \in \mathbb{N}}, \{v_j\}_{j \in \mathbb{N}}$ in H such that

$$S(\xi) = \sum_{j=1}^{\infty} \sigma_j \langle \xi, v_j \rangle u_j$$

for all $\xi \in H$.

2c (weight 10 points)

Determine $\ker(S)$ and $\ker(S^*)$. Does S satisfies the Fredholm alternative?

(Continued on page 3.)

Problem 3 (weight 20 points)

Let X and Y be normed spaces (over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$).

3a (weight 10 points)

Assume $\{y_n\}_{n \in \mathbb{N}}$ is a sequence in Y such that $\{\varphi(y_n)\}_{n \in \mathbb{N}}$ is a bounded sequence in $\mathbb{K}$ for every $\varphi \in Y^*$. Show that $\{y_n\}_{n \in \mathbb{N}}$ is bounded in Y .

Hint. You have to use the principle of uniform boundedness in a suitable way. If you are not able to answer this problem, you have the option to state what this principle says (and get up to 5 points).

3b (weight 10 points)

Assume $T : X \rightarrow Y$ is a linear operator such that $\varphi \circ T \in X^*$ for every $\varphi \in Y^*$. Show that T is bounded.

Hint. Assume that T is unbounded and show that this leads to a contradiction.

Problem 4 (weight 10 points)

Let T be a compact operator on an infinite dimensional Hilbert space H (over $\mathbb{R}$ or $\mathbb{C}$) and let $\{u_n\}_{n \in \mathbb{N}}$ be an orthonormal sequence in H .

Show that $\lim_{n \rightarrow \infty} \|T(u_n)\| = 0$.

Good luck!