

Notes for MAT4410

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Contents

Contents	i
1 On normed spaces and bounded linear operators	1
1.1 Preliminaries	1
1.2 Aspects of finite dimensionality	5
1.3 Extension by density and continuity	9
1.4 Exercises	12
2 On Hilbert spaces and bounded linear operators	15
2.1 Inner product spaces and Hilbert spaces	15
2.2 Geometry in Hilbert spaces	18
2.3 Orthonormal bases in Hilbert spaces	24
2.4 Adjoint operators	30
2.5 Self-adjoint operators	37
2.6 Unitary operators	40
2.7 Exercises	44
3 On Banach spaces and bounded linear operators	53
3.1 Quotient spaces	53
3.2 The Baire category theorem	58
3.3 The principle of uniform boundedness	60
3.4 The open mapping theorem	63
3.5 The closed graph theorem	67
3.6 The second dual, reflexive spaces, and duals of quotient spaces	69
3.7 Exercises	78
4 On compact operators	85
4.1 Introduction to compact operators between normed spaces .	85

Contents

4.2	Compact operators on Hilbert spaces	90
4.3	The spectral theorem for a compact self-adjoint operator . .	96
4.4	The Fredholm Alternative	103
4.5	Exercises	111
5	An introduction to Sturm-Liouville theory	117
5.1	Regular Sturm-Liouville systems	117
5.2	A second order differential equation	119
5.3	Some spectral properties	122
5.4	A special case	124
5.5	The general case	131
5.6	Exercises	133

CHAPTER 1

On normed spaces and bounded linear operators

1.1 Preliminaries

In this subsection we fix some notation and review some of the concepts and results that we will need. These are usually covered in undergraduate courses in real analysis, as for example in the course MAT2400 and MAT3400/4400 at UiO. The reader may consult the book of T. Lindstrøm, *Spaces: an introduction to real analysis* (AMS 2017) which is used in MAT2400. One may also have a look at the book (or notes) for MAT3400/4400, and/or chapters 3, 4 and 11 in J.C. Robinson's book *An introduction to functional analysis* (Cambridge University Press 2020), which can be used as background literature in this course; when making a reference to this book, we will denote it by [R].

Throughout these notes, $\mathbb{K}$ will denote either $\mathbb{R}$ (the real numbers) or $\mathbb{C}$ (the complex numbers). If X, Y are sets, we let $X \times Y$ denote their Cartesian product, i.e.,

$$X \times Y = \{(x, y) : x \in X, y \in Y\}.$$

A metric space (X, d) is called *complete* when every Cauchy sequence in (X, d) is convergent.

Definition 1.1.1. A *normed space* $(X, \|\cdot\|)$ over $\mathbb{K}$ is a vector space X over $\mathbb{K}$ which is equipped with a norm $\|\cdot\|$. We recall that X is then a metric space with respect to the metric given by $d(x, y) = \|x - y\|$ for $x, y \in X$. We will only consider normed spaces over $\mathbb{K}$ in these notes, and we will often just write X to denote such a normed space, assuming tacitly that some norm on X is given.

1. On normed spaces and bounded linear operators

When there is no danger of confusion we will denote the closed unit ball in X with center at 0 by X_1 . Thus,

$$X_1 := \{x \in X : \|x\| \leq 1\}.$$

Definition 1.1.2. If $(X, \|\cdot\|)$ is a normed space, and $\|\cdot\|'$ is also a norm on X , we say that $\|\cdot\|$ and $\|\cdot\|'$ are *equivalent* when there exist positive real numbers K and L such that

$$\|x\|' \leq K \|x\| \quad \text{and} \quad \|x\| \leq L \|x\|' \quad \text{for all } x \in X.$$

When $\|\cdot\|$ and $\|\cdot\|'$ are equivalent, it is clear that a sequence $\{x_n\}_{n=1}^{\infty}$ in X converges to $x \in X$ w.r.t. $\|\cdot\|$ if and only if it converges to $x \in X$ w.r.t. $\|\cdot\|'$.

Definition 1.1.3. Assume $\{x_n\}_{n=1}^{\infty}$ is a sequence in a normed space $(X, \|\cdot\|)$. We say that the series $\sum_{n=1}^{\infty} x_n$ is *convergent in X* if there is some $x \in X$ such that

$$\lim_{N \rightarrow \infty} \left\| x - \sum_{n=1}^N x_n \right\| = 0,$$

in which case we say that $\sum_{n=1}^{\infty} x_n$ converges to x (w.r.t. $\|\cdot\|$), and also write $x = \sum_{n=1}^{\infty} x_n$.

Definition 1.1.4. When a normed space $(X, \|\cdot\|)$ is complete with respect to the associated metric given by

$$d(x, y) = \|x - y\|$$

for all $x, y \in X$, we say that X is a *Banach space*.

To check that a normed space is a Banach space, the following result is often useful:

Theorem 1.1.5. *Let $(X, \|\cdot\|)$ be a normed space. Then X is a Banach space if and only if every absolutely convergent series in X is convergent in X , that is, if and only if the following condition holds :*

Whenever $\sum_{n=1}^{\infty} x_n$ is a series in X such that $\sum_{n=1}^{\infty} \|x_n\| < \infty$, then $\sum_{n=1}^{\infty} x_n$ is convergent in X .

Remark 1.1.6. It is good to know that if X is a normed space, then we can always form its *completion*; this means that whenever needed, we can assume that X sits as a dense subspace of a Banach space $\widetilde{X}$ where the norm of $\widetilde{X}$ extends the norm on X . You can read about how this can be done in [R, section 7.2] if you like. A more elegant way to construct $\widetilde{X}$ will be described in Remark 3.6.2.

Definition 1.1.7. Assume that X and Y are both vector spaces over $\mathbb{K}$. Then a map $T : X \rightarrow Y$ is called a *linear operator* if we have

$$T(\lambda_1 x_1 + \lambda_2 x_2) = \lambda_1 T(x_1) + \lambda_2 T(x_2)$$

for all $\lambda_1, \lambda_2 \in \mathbb{K}$ and all $x_1, x_2 \in X$.

We let $\mathcal{L}(X, Y)$ denote the set of all linear operators from X to Y . One readily checks that $\mathcal{L}(X, Y)$ is a vector space over $\mathbb{K}$ with respect to the operations defined by

$$(S + T)(x) = S(x) + T(x), \quad (\lambda T)(x) = \lambda T(x)$$

for $S, T \in \mathcal{L}(X, Y)$, $\lambda \in \mathbb{K}$ and $x \in X$. We also set $\mathcal{L}(X) := \mathcal{L}(X, X)$. We let $I_X \in \mathcal{L}(X)$ denote the *identity map from X into itself*, that is, $I_X(x) = x$ for all $x \in X$. We sometimes write I instead of I_X if no confusion is possible.

Definition 1.1.8. Assume that X and Y are both normed spaces over $\mathbb{K}$. Then a linear operator $T : X \rightarrow Y$ is called *bounded* if there exists some real number $M > 0$ such that

$$\|T(x)\| \leq M \|x\| \quad \forall x \in X,$$

or, equivalently, such that $\|T(x)\| \leq M$ for all $x \in X_1$.

Proposition 1.1.9. Assume that X and Y are both normed spaces over $\mathbb{K}$ and let $T \in \mathcal{L}(X, Y)$. Then the following conditions are equivalent:

- (a) T is bounded.
- (b) T is uniformly continuous on X .
- (c) T is continuous on X .
- (d) T is continuous at $x = 0$.

We will denote the set of all bounded linear operators from X to Y by $\mathcal{B}(X, Y)$. We follow tradition here and use the qualifying adjective “bounded”, although we could equally well have used “continuous” instead. One readily checks that $\mathcal{B}(X, Y)$ is a subspace of $\mathcal{L}(X, Y)$. We also set $\mathcal{B}(X) = \mathcal{B}(X, X)$.

Proposition 1.1.10. Assume that X and Y are both normed spaces over $\mathbb{K}$. For $T \in \mathcal{B}(X, Y)$, set

$$\|T\| := \sup \{ \|T(x)\| : x \in X, \|x\| \leq 1 \} < \infty.$$

1. On normed spaces and bounded linear operators

Then the map $T \mapsto \|T\|$ is a norm on $\mathcal{B}(X, Y)$, called the operator norm, making $\mathcal{B}(X, Y)$ a normed space over $\mathbb{K}$. Moreover, we have

$$\|T\| = \sup \{ \|T(x)\| : x \in X, \|x\| = 1 \} \quad (\text{when } X \neq \{0\}),$$

and

$$\|T(x)\| \leq \|T\| \|x\| \quad \text{for all } x \in X.$$

Theorem 1.1.11. Assume that X is a normed space and Y is a Banach space (both over $\mathbb{K}$). Then $\mathcal{B}(X, Y)$ is Banach space. In particular, $\mathcal{B}(X)$ is a Banach space whenever X is a Banach space.

An immediate consequence of this theorem is that $\mathcal{B}(X, \mathbb{K})$ is a Banach space whenever X is normed space over $\mathbb{K}$. Elements of $\mathcal{L}(X, \mathbb{K})$ are called *linear functionals*. Thus $\mathcal{B}(X, \mathbb{K})$ consists of the *bounded linear functionals on X* ; it is usually called the *dual space of X* and denoted by X^* . An important result about X^* is the Hahn-Banach extension theorem.

Theorem 1.1.12 (Hahn-Banach). Let X be a normed space (over $\mathbb{K}$) and p be a seminorm on X . Assume X_0 is a subspace of X and φ_0 is a linear functional on X_0 such that $|\varphi_0(x)| \leq p(x)$ for all $x \in X_0$. Then φ_0 can be extended to a linear functional φ on X such that $|\varphi(x)| \leq p(x)$ for all $x \in X$.

We also recall the following useful consequences of the Hahn-Banach extension theorem.

Corollary 1.1.13. Assume X_0 is a subspace of a normed space X (over $\mathbb{K}$) and $\varphi_0 \in (X_0)^*$. Then there exists some $\varphi \in X^*$ which extends φ_0 and satisfies that $\|\varphi\| = \|\varphi_0\|$.

Corollary 1.1.14. Assume X is a normed space (over $\mathbb{K}$) and $x \in X, x \neq 0$. Then there exists some $\varphi \in X^*$ such that $\|\varphi\| = 1$ and $\varphi(x) = \|x\|$.

Definition 1.1.15. A map $T : X \rightarrow Y$ between two vector spaces over $\mathbb{K}$ is called a (*vector space*) *isomorphism* if $T \in \mathcal{L}(X, Y)$ and T is bijective (that is, T is both one-to-one and onto). It is then easy to check that the inverse map of T , $T^{-1} : Y \rightarrow X$, is linear, i.e., $T^{-1} \in \mathcal{L}(Y, X)$.

Definition 1.1.16. Assume that X and Y are normed spaces over $\mathbb{K}$. A map $T : X \rightarrow Y$ is called an *isomorphism of normed spaces* if T is a (vector space) isomorphism such that both T and T^{-1} are bounded.

1.2. Aspects of finite dimensionality

Definition 1.1.17. Assume that X is a normed space and $T \in \mathcal{B}(X)$. Then we say that T is *invertible in $\mathcal{B}(X)$* if T is an isomorphism of normed spaces. In other words, an operator $T \in \mathcal{B}(X)$ is invertible in $\mathcal{B}(X)$ if T is bijective and $T^{-1} \in \mathcal{B}(X)$.

Proposition 1.1.18. Let X, Y, Z be normed spaces over $\mathbb{K}$, and let $T \in \mathcal{B}(X, Y)$, $S \in \mathcal{B}(Y, Z)$. Set $ST := S \circ T : X \rightarrow Z$. Then $ST \in \mathcal{B}(X, Z)$ and

$$\|ST\| \leq \|S\| \|T\|.$$

Corollary 1.1.19. Assume that X is a normed space and $S \in \mathcal{B}(X)$. For each $n \in \mathbb{N}$, let $S^n := S \cdots S$ denote the product of S with itself n times. Then $S^n \in \mathcal{B}(X)$ and

$$\|S^n\| \leq \|S\|^n.$$

Note that by setting $S^0 := I_X$, this formula also holds when $n = 0$.

1.2 Aspects of finite dimensionality

Finite-dimensionality of a normed space has some strong consequences. A useful fact that you should have learned in MAT2400 (or a similar course) is the following result.

Theorem 1.2.1. If X is a finite-dimensional vector space over $\mathbb{K}$, then all norms on X are equivalent.

This is Theorem 5.1 in [R]. We recommend that you read again the proof if you don't remember how it goes. A crucial fact used in this proof is that $\mathbb{K}^n$ satisfies the *Heine-Borel property* with respect to the Euclidean norm $\|\cdot\|_2$: a subset S of $\mathbb{K}^n$ is compact (w.r.t. the metric associated with $\|\cdot\|_2$) if and only if S is closed and bounded. We recall that $\|\cdot\|_2$ is given by

$$\|x\|_2 = \left(|x_1|^2 + \cdots + |x_n|^2\right)^{1/2} \quad \text{for } x = (x_1, \dots, x_n) \in \mathbb{K}^n,$$

and that a subset S of a metric space is called *compact* if every sequence in S has a subsequence converging to a point in S . It will be useful for us to know that this characterization of compactness holds in any finite-dimensional normed space. To show this, we will need the following lemma.

Lemma 1.2.2. Let X and Y be finite-dimensional normed spaces over $\mathbb{K}$. Assume that X and Y are isomorphic as vector spaces and let $T \in \mathcal{L}(X, Y)$ be an isomorphism. Then T is an isomorphism of normed spaces.

1. On normed spaces and bounded linear operators

Proof. We have to show that T and T^{-1} are bounded. We denote the respective norms on X and Y by the same symbol $\|\cdot\|$. For $x \in X$ we set

$$\|x\|_T := \|T(x)\|.$$

Since T is 1-1, it is easy to check that the map $x \mapsto \|\cdot\|_T$ is a norm on X . Since X is finite-dimensional, $\|\cdot\|_T$ is equivalent to $\|\cdot\|$. In particular, this means that there exists some $L > 0$ such that

$$\|T(x)\| = \|x\|_T \leq L \|x\| \quad \text{for all } x \in X,$$

which shows that T is bounded. Similarly, by considering the norm on Y given by $\|y\|_{T^{-1}} := \|T^{-1}(y)\|$ for $y \in Y$, one deduces that T^{-1} is also bounded. ■

Theorem 1.2.3. *Let X be a finite-dimensional normed space over $\mathbb{K}$. Then a subset S of X is compact if and only if S is closed and bounded.*

Proof. Since a compact subset of a metric space is always closed and bounded, we only have to show the reverse implication. So let $S \subseteq X$ be closed and bounded. We must show that S is compact. If $X = \{0\}$, this is obviously true, so we may assume that $m := \dim(X) \geq 1$. Let then $T : X \rightarrow \mathbb{K}^m$ denote the coordinate map w.r.t. some basis for X . Lemma 1.2.2 gives that T is an isomorphism of normed spaces. Set $S' := T(S) \subseteq \mathbb{K}^m$. Then S' is bounded (since T is bounded). Moreover, S' is closed. Indeed, as $S' = (T^{-1})^{-1}(S)$, this follows from the continuity of T^{-1} . By the Heine-Borel property of $\mathbb{K}^m$, we can conclude that S' is compact. As $S = T^{-1}(S')$ and T^{-1} is continuous, this implies that S is compact, as desired. ■

Since the closed unit ball of any normed space is closed and bounded, we get as a corollary that the closed unit ball of a finite-dimensional normed space is compact. Quite remarkably, the converse is also true, as we will soon see, cf. Theorem 1.2.6.

A property which is automatically satisfied for a finite-dimensional normed space is completeness:

Proposition 1.2.4. *Let X be a finite-dimensional normed space. Then X is a Banach space.*

Proof. We may clearly assume that $X \neq \{0\}$. To show that X is complete, we let $\{x_n\}_{n \in \mathbb{N}}$ be a Cauchy sequence in X and have to prove that it is convergent. As in the proof of Proposition 1.2.3, we can pick an isomorphism of normed spaces $T : X \rightarrow \mathbb{K}^m$, where $m = \dim(X)$. For each $n \in \mathbb{N}$, set

1.2. Aspects of finite dimensionality

$y_n := T(x_n)$. Since $\|y_n - y_k\|_2 = \|T(x_n - x_k)\|_2 \leq \|T\| \|x_n - x_k\|$ for all $k, n \in \mathbb{N}$, we see that $\{y_n\}_{n \in \mathbb{N}}$ is a Cauchy sequence in $\mathbb{K}^m$. Since $\mathbb{K}^m$ is complete, there exists $y \in \mathbb{K}^m$ such that $\|y_n - y\|_2 \rightarrow 0$ as $n \rightarrow \infty$. Set $x := T^{-1}(y) \in X$. Then we get

$$\|x_n - x\| = \|T^{-1}(y_n - y)\| \leq \|T^{-1}\| \|y_n - y\|_2 \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Thus, $\{x_n\}_{n \in \mathbb{N}}$ is convergent, as desired. ■

Corollary 1.2.5. *Assume M is a finite-dimensional subspace of a normed space X . Then M is closed in X .*

Proof. Assume $\{x_n\}_{n \in \mathbb{N}} \subseteq M$ converges to $x \in X$. We have to show that $x \in M$. As M is complete by Proposition 1.2.4, and $\{x_n\}_{n \in \mathbb{N}}$ is a Cauchy sequence in M , it follows that $\{x_n\}_{n \in \mathbb{N}}$ converges to some $y \in M$. Thus we get that $x = \lim_{n \rightarrow \infty} x_n = y \in M$. ■

Theorem 1.2.6. *Let X be a normed space over $\mathbb{K}$. Then X is finite-dimensional if and only if the closed unit ball X_1 is compact.*

Proof. We have seen earlier that the implication $(\Rightarrow)$ holds. So we only have to prove $(\Leftarrow)$. We will use the following claim (which is often called *Riesz's lemma*):

Assume M is a closed subspace of X such that $M \neq X$. Then there exists $x \in X$ such that $\|x\| = 1$ and

$$\frac{1}{2} \leq \|x - m\| \quad \text{for all } m \in M. \tag{1.2.1}$$

To prove this claim, choose $x_0 \in X \setminus M$ and set $d := \inf_{m \in M} \|x_0 - m\|$ (so d is the distance from x_0 to M).

Note that $d > 0$ (because if $d = 0$, we would have $x_0 \in \overline{M} = M$, giving a contradiction). As $d < 2d$ we can pick $m_0 \in M$ such that

$$t_0 := \|x_0 - m_0\| \leq 2d.$$

Further, since $t_0 = \|x_0 - m_0\| \geq d > 0$, we can set

$$x := \frac{1}{t_0}(x_0 - m_0).$$

Then $\|x\| = 1$. Moreover, for $m \in M$, we have

$$\|x - m\| = \left\| \frac{1}{t_0}(x_0 - m_0) - m \right\| = \frac{1}{t_0} \|x_0 - (m_0 + t_0 m)\|.$$

1. On normed spaces and bounded linear operators

Since M is a subspace, $m_0 + t_0 m \in M$. So $\|x_0 - (m_0 + t_0 m)\| \geq d$. Thus we get that

$$\|x - m\| \geq \frac{1}{t_0} d \geq \frac{1}{2}.$$

This proves that (1.2.1) holds, showing the claim.

We can now proceed with the proof of $(\Leftarrow)$ in Theorem 1.2.6. We argue contrapositively. Assume that X is not finite-dimensional. We will then show that X_1 is not compact.

Pick first $x_1 \in X$ such that $\|x_1\| = 1$. Then $M_1 := \text{span}\{x_1\}$ is a finite-dimensional subspace of X , so it is closed by Corollary 1.2.5 and $M_1 \neq X$ (since X is not finite-dimensional). Hence, using the claim, we can find $x_2 \in X$ such that $\|x_2\| = 1$ and $\|x_2 - x_1\| \geq \frac{1}{2}$.

Set now $M_2 := \text{span}\{x_1, x_2\}$. Arguing similarly, we can find $x_3 \in X$ such that $\|x_3\| = 1$ and $\|x_3 - x_k\| \geq \frac{1}{2}$ for $k = 1, 2$.

Proceeding inductively we obtain a sequence $\{x_n\}_{n \in \mathbb{N}}$ in X_1 such that $1/2 \leq \|x_n - x_k\|$ for all $1 \leq k < n$. It is then clear that no subsequence of $\{x_n\}_{n \in \mathbb{N}}$ can be Cauchy. In particular, $\{x_n\}_{n \in \mathbb{N}}$ has no convergent subsequence. Thus, X_1 is not compact, as we wanted to show. $\blacksquare$

We recall that a vector space is said to be *infinite-dimensional* if it is not finite-dimensional. We note that Theorem 1.2.6 gives that if X is an infinite-dimensional normed space, then X_1 is *not* compact. In particular, this implies that an infinite-dimensional normed space never has the Heine-Borel property.

Finite dimensionality has also some impact on linear operators. We first look at a familiar case.

Example 1.2.7. Let $m, n \in \mathbb{N}$ and let $T \in \mathcal{L}(\mathbb{K}^n, \mathbb{K}^m)$. Then T is bounded.

Indeed, let $A = [a_{i,j}]$ denote the standard matrix of T . Then for $x = (x_1, \dots, x_n) \in \mathbb{K}^n$ we have $T(x) = (F_1(x), \dots, F_m(x))$, where $F_i(x) := \sum_{j=1}^n a_{i,j} x_j$ for each $i = 1, \dots, m$. Since each component F_i is clearly a continuous function from $\mathbb{K}^n$ to $\mathbb{K}$, we get that the linear map $T : \mathbb{K}^n \rightarrow \mathbb{K}^m$ is continuous, hence bounded.

More generally, we have:

Proposition 1.2.8. *Let X and Y be normed spaces and let $T \in \mathcal{L}(X, Y)$. Assume that X is finite-dimensional. Then T is bounded.*

Proof. We may clearly assume that both X and Y are different from $\{0\}$. Moreover, by replacing Y with $T(X)$ if necessary, we may also assume that Y

1.3. Extension by density and continuity

is finite-dimensional. Set $n = \dim(X)$, $m = \dim(Y)$, and let $C : X \rightarrow \mathbb{K}^n$, $D : Y \rightarrow \mathbb{K}^m$ be isomorphisms, which are then necessarily isomorphisms of normed spaces by Lemma 1.2.2. The composition $T' := D \circ T \circ C^{-1}$ is then a linear map from $\mathbb{K}^n$ to $\mathbb{K}^m$, hence it is bounded by the previous example. It follows that $T = D^{-1} \circ T' \circ C$, being the composition of bounded maps, is bounded. ■

Note that the above result is not true in general if we instead assume that Y is finite-dimensional, even in the case where $Y = \mathbb{K}$: a linear functional $\ell : X \rightarrow \mathbb{K}$ may be unbounded when X is an infinite-dimensional normed space. For an example, see Exercise 1.1.

1.3 Extension by density and continuity

This short section is devoted to a very useful principle in linear analysis, often called *the principle of extension by density and continuity*. We will need the following elementary lemma, which you may have seen before.

Lemma 1.3.1. *Assume that X and Y are metric spaces and f, g are continuous maps from X to Y which agree on a dense subset X_0 of X . Then $f = g$.*

Proof. Let $x \in X$. Since X_0 is dense in X , there exists a sequence $\{x_n\}_{n \in \mathbb{N}}$ in X_0 which converges to x . By continuity of f and g , we get

$$f(x) = \lim_n f(x_n) = \lim_n g(x_n) = g(x). \quad \blacksquare$$

Theorem 1.3.2. *Assume that X is a normed space and Y is a Banach space (both over $\mathbb{K}$). Assume also that X_0 is a dense subspace of X , Y_0 is a subspace of Y and $T_0 \in \mathcal{B}(X_0, Y_0)$.*

Then T_0 extends in a unique way to an operator $T \in \mathcal{B}(X, Y)$. It satisfies that $\|T\| = \|T_0\|$.

Proof. Let $x \in X$. Since X_0 is dense in X , there exists a sequence $\{x_n\}_{n \in \mathbb{N}}$ in X_0 such that $\|x - x_n\| \rightarrow 0$ as $n \rightarrow \infty$. In particular, $\{x_n\}_{n \in \mathbb{N}}$ is a Cauchy sequence in X_0 . We claim that $\{T_0(x_n)\}_{n \in \mathbb{N}}$ is a Cauchy sequence in Y . Indeed, let $\varepsilon > 0$, and choose $N \in \mathbb{N}$ such that

$$\|x_m - x_n\| < \varepsilon / \|T_0\| \quad \text{for all } m, n \geq N.$$

Then, for all $m, n \geq N$, we get

$$\|T_0(x_m) - T_0(x_n)\| = \|T_0(x_m - x_n)\| = \|T_0\| \|x_m - x_n\| < \varepsilon,$$

1. On normed spaces and bounded linear operators

as desired.

Since Y is complete, we can conclude that there exists some $y \in Y$ such that $\lim_n T_0(x_n) = y$. Note that y only depends on x . Indeed, assume $\{x'_n\}_{n \in \mathbb{N}}$ is another sequence in X_0 converging to x . Then the sequence $x_1, x'_1, x_2, x'_2, \dots, x_n, x'_n, \dots$ in X_0 also converges to x , so, arguing as above, we get that there exists some $z \in Y$ such that the sequence $T_0(x_1), T_0(x'_1), T_0(x_2), T_0(x'_2), \dots, T_0(x_n), T_0(x'_n), \dots$ converges to z . This implies that

$$\lim_n T_0(x'_n) = z = \lim_n T_0(x_n) = y.$$

Hence it makes sense to define $T(x) := y$. Doing this for every $x \in X$, we get a map $T : X \rightarrow Y$, and it is easy to check that T is linear, so we leave this as an exercise.

Next, we show that T is bounded. Let $x \in X$ and pick $\{x_n\}_{n \in \mathbb{N}}$ in X_0 converging to x . As $T(x) = \lim_n T_0(x_n)$ and $\|T_0(x_n)\| \leq \|T_0\| \|x_n\|$ for all $n \in \mathbb{N}$, we get

$$\|T(x)\| = \lim_n \|T_0(x_n)\| \leq \|T_0\| \lim_n \|x_n\| = \|T_0\| \|x\|.$$

It follows that $T \in \mathcal{B}(X, Y)$ with $\|T\| \leq \|T_0\|$.

Further, T is an extension of T_0 . Indeed, let $x \in X_0$. Then set $x_n := x$ for all $n \in \mathbb{N}$. Since $\{x_n\}_{n \in \mathbb{N}}$ is a sequence in X_0 converging to x , we get that

$$T(x) = \lim_n T_0(x_n) = T_0(x).$$

The uniqueness of T as an extension of T_0 is immediate from Lemma 1.3.1. Finally, we have

$$\begin{aligned} \|T_0\| &= \sup\{\|T_0(x)\| : x \in X_0, \|x\| \leq 1\} \\ &= \sup\{\|T(x)\| : x \in X_0, \|x\| \leq 1\} \\ &\leq \sup\{\|T(x)\| : x \in X, \|x\| \leq 1\} = \|T\| \leq \|T_0\|. \end{aligned}$$

Thus, $\|T\| = \|T_0\|$, as desired. ■

Remark 1.3.3. The conclusion of Theorem 1.3.2 is not necessarily true if Y is a normed space which is not complete (cf. Exercise 1.3). □

An interesting special case of Theorem 1.3.2 is when T_0 is an isometry. We recall that a linear map between normed spaces is called an *isometry* when it is norm-preserving. A linear isometry is clearly bounded.

Corollary 1.3.4. *Let the assumptions be as in Theorem 1.3.2, but assume in addition that the map $T_0 : X_0 \rightarrow Y_0$ is an isometry. Then the unique extension of T_0 to an operator T in $\mathcal{B}(X, Y)$ is also an isometry.*

1.3. Extension by density and continuity

Proof. Theorem 1.3.2 guarantees that T_0 extends in a unique way to $T \in \mathcal{B}(X, Y)$. Let $x \in X$ and pick a sequence $\{x_n\}_{n \in \mathbb{N}}$ in X_0 converging to x . We then have $T(x) = \lim_n T_0(x_n)$, so we get

$$\|T(x)\| = \lim_n \|T_0(x_n)\| = \lim_n \|x_n\| = \|x\|. \quad \blacksquare$$

Using Corollary 1.3.4, it can be shown that the completion of a (non-complete) normed space is unique up to isometric isomorphism (cf. Exercise 1.5). We also record an important particular case of Theorem 1.3.2.

Corollary 1.3.5. *Assume that X is a Banach space over $\mathbb{K}$ and X_0 is a dense subspace of X . Then every $T_0 \in \mathcal{B}(X_0)$ extends in a unique way to an operator $T \in \mathcal{B}(X)$, which satisfies that $\|T\| = \|T_0\|$.*

Example 1.3.6. Let $a, b \in \mathbb{R}$, $a < b$. Equip the space $C([a, b])$ of all continuous complex functions on $[a, b]$ with the norm $\|f\|_2 = (\int_a^b |f(s)|^2 ds)^{1/2}$.

Let $K : [a, b] \times [a, b] \rightarrow \mathbb{C}$ be a continuous function.¹ We can then associate to K an *integral operator* $T_K : C([a, b]) \rightarrow C([a, b])$ as follows.

Let $f \in C([a, b])$. Since the assignment $t \mapsto K(s, t)f(t)$ gives a continuous function on $[a, b]$ for each $s \in [a, b]$, we may define a function $T_K(f) : [a, b] \rightarrow \mathbb{C}$ by

$$[T_K(f)](s) = \int_a^b K(s, t) f(t) dt \quad \text{for all } s \in [a, b].$$

We leave it as an exercise to verify that $T_K(f)$ is a continuous function on $[a, b]$, satisfying

$$\|T_K(f)\|_2 \leq \left(\int_a^b \int_a^b |K(s, t)|^2 ds dt \right)^{1/2} \|f\|_2.$$

As the map $f \mapsto T_K(f)$ is clearly linear, it follows that T_K is a bounded linear operator from $C([a, b])$ into itself.

Let now $L^2([a, b])$ denote the complex L^2 -space associated with the measure space $([a, b], \mathcal{A}, \mu)$, where μ is the Lebesgue measure on the σ -algebra $\mathcal{A}$ of all Lebesgue measurable subsets of $[a, b]$.

Recall that $C([a, b])$ naturally sits as a dense closed subspace of $L^2([a, b])$. We therefore get from Corollary 1.3.5 that T_K has a unique extension to a bounded operator on $L^2([a, b])$, also denoted by T_K . The function K is usually called the *kernel* of the integral operator T_K . We will come back to integral operators later.

¹Of course, we consider here the square $[a, b] \times [a, b]$ as a metric space w.r.t. the Euclidean metric inherited from $\mathbb{R}^2$.

1.4 Exercises

Exercise 1.1. Recall that $\ell^\infty(\mathbb{N})$ denotes the space of all bounded functions from $\mathbb{N}$ into $\mathbb{C}$. Recall also that it is a normed space over $\mathbb{C}$ w.r.t. the norm $\|f\|_\infty = \sup_{n \in \mathbb{N}} |f(n)|$. It is often denoted by ℓ^∞ and thought of as the space of bounded complex sequences (under the identification $f \mapsto \{f(j)\}_{j \in \mathbb{N}}$).

Let X be the subspace of $\ell^\infty(\mathbb{N})$ given by

$$X = \{f : \mathbb{N} \rightarrow \mathbb{C} : f(n) = 0 \text{ for all but finitely many } n\}.$$

a) Show that X is infinite-dimensional.

b) Consider X as a normed space w.r.t. $\|\cdot\|_\infty$ and let $\ell : X \rightarrow \mathbb{C}$ be defined by

$$\ell(f) = \sum_{n=1}^{\infty} f(n)$$

for all $f \in X$. Clearly, $\ell \in \mathcal{L}(X, \mathbb{C})$. Show that ℓ is unbounded.

Exercise 1.2. Let $\mathcal{P}_{\mathbb{R}}$ denote the real vector space consisting of all polynomials in one real variable with real coefficients. For $p \in \mathcal{P}_{\mathbb{R}}$, set

$$\|p\| := \sup_{t \in [0,1]} |p(t)|.$$

a) Explain why $p \mapsto \|p\|$ gives a well-defined norm on $\mathcal{P}_{\mathbb{R}}$.

b) Define a linear operator $D : \mathcal{P}_{\mathbb{R}} \rightarrow \mathcal{P}_{\mathbb{R}}$ by

$$D(p) = p' \quad (\text{the derivative of } p).$$

Show that D is unbounded. Conclude that $\mathcal{P}_{\mathbb{R}}$ is infinite-dimensional.

Exercise 1.3. Let X be a Banach space having a dense subspace X_0 which is not complete. Consider the identity map $I_0 : X_0 \rightarrow X_0$. Show that I_0 does not have an extension to a bounded linear map from X into X_0 .

Exercise 1.4. Assume that X is a normed space and Y is a Banach space (both over $\mathbb{K}$), and let $\{T_k\}_{k \in \mathbb{N}}$ be a sequence in $\mathcal{B}(X, Y)$ which is *uniformly bounded* in the sense that $M := \sup_{k \in \mathbb{N}} \|T_k\| < \infty$.

Moreover, assume that there exists a dense subset S of X such that $\{T_k(x)\}_{k \in \mathbb{N}}$ converges in Y for every $x \in S$.

Show that there exists $T \in \mathcal{B}(X, Y)$ such that

$$T(x) = \lim_k T_k(x) \quad \text{for all } x \in X.$$

Exercise 1.5. Assume X_0 is a normed space over $\mathbb{K}$ and let (X, i) denote a *completion* of X_0 , that is, X is a Banach space and $i : X_0 \rightarrow X$ is a linear isometry such that $i(X_0)$ is dense in X . (As mentioned in Remark 1.1.6, such a completion always exists.)

Show that (X, i) is unique up to isometric isomorphism, meaning that the following holds: if (X', i') is another completion of X_0 , then there exists an isometric isomorphism $U : X \rightarrow X'$ such that $i' = U \circ i$.

Exercise 1.6. Provide the details missing in Example 1.3.6.

CHAPTER 2

On Hilbert spaces and bounded linear operators

2.1 Inner product spaces and Hilbert spaces

We recall that $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Inner product spaces over $\mathbb{K}$ should be familiar from previous courses. In this section we give a unified review of the most basic results about such spaces.

Definition 2.1.1. An *inner product space over $\mathbb{K}$* is a vector space X over $\mathbb{K}$ which is equipped with an inner product $\langle \cdot, \cdot \rangle : X \times X \rightarrow \mathbb{K}$. This means that for $x, y, z \in X$ and $\lambda \in \mathbb{K}$ we have:

- i) $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$,
- ii) $\langle \lambda x, y \rangle = \lambda \langle x, y \rangle$,
- iii) $\langle y, x \rangle = \overline{\langle x, y \rangle}$,
- iv) $\langle x, x \rangle \geq 0$,
- v) $\langle x, x \rangle = 0$ if and only if $x = 0$.

Remark 2.1.2. a) Properties i) and ii) say that the inner product is linear in the first variable.

b) When $\mathbb{F} = \mathbb{R}$, property iii) says that the inner product is symmetric, i.e., $\langle y, x \rangle = \langle x, y \rangle$; combining i) and ii) with iii), we then get that the inner product is also linear in the second variable.

c) When $\mathbb{F} = \mathbb{C}$, we get that the inner product is *conjugate-linear* in the second variable; this means that we have

$$\langle x, y + z \rangle = \langle x, y \rangle + \langle x, z \rangle \quad \text{and} \quad \langle x, \lambda y \rangle = \overline{\lambda} \langle x, y \rangle.$$

2. On Hilbert spaces and bounded linear operators

Some authors prefer to use inner products that are linear in the second variable and conjugate-linear in the first variable. This is common in textbooks related to physics or mathematical physics. As one can go from one type to the other by setting $\langle x, y \rangle' := \langle y, x \rangle$, it is mainly a matter of taste which convention one chooses to use. $\square$

Example 2.1.3. The standard example of an inner product space over $\mathbb{K}$ is $\mathbb{K}^n$, where $n \in \mathbb{N}$, equipped with

$$\langle \mathbf{x}, \mathbf{y} \rangle := \sum_{j=1}^n x_j \overline{y_j}, \quad \mathbf{x} = (x_1, \dots, x_n), \mathbf{y} = (y_1, \dots, y_n) \in \mathbb{K}^n. \quad \square$$

In the sequel, by an inner product space, we will always mean an inner product space over $\mathbb{K}$. An inequality of fundamental importance is:

Theorem 2.1.4 (*The Cauchy-Schwarz inequality*). *Let X be an inner product space. Then we have*

$$|\langle x, y \rangle| \leq \|x\| \|y\| \quad (2.1.1)$$

for all $x, y \in X$, with equality if and only if x and y are linearly dependent.

If X is an inner product space, then using the Cauchy-Schwarz inequality, one deduces that $\|x\| := \langle x, x \rangle^{1/2}$ gives a norm on X . Thus, X is then a normed space, and its norm is easily seen to satisfy the *parallelogram law*, that is, for all $x, y \in X$ we have

$$\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2. \quad (2.1.2)$$

Moreover, the so-called *polarization identities* are sometimes useful:

$$\langle x, y \rangle = \frac{1}{4} (\|x + y\|^2 - \|x - y\|^2) \quad \text{if } \mathbb{K} = \mathbb{R}, \quad (2.1.3)$$

$$\langle x, y \rangle = \frac{1}{4} \sum_{k=0}^3 i^k \|x + i^k y\|^2 \quad \text{if } \mathbb{K} = \mathbb{C}. \quad (2.1.4)$$

Definition 2.1.5. Let X be an inner product space. If $x, y \in X$, then x and y are said to be *orthogonal* (to each other) when $\langle x, y \rangle = 0$. If U, V are nonempty subsets of X , we write $U \perp V$ when $\langle u, v \rangle = 0$ for all $u \in U, v \in V$. A nonempty subset S of X is called *orthogonal* if x and y are orthogonal for all $x, y \in S$ such that $x \neq y$. Moreover, S is called *orthonormal* if S is orthogonal and $\|x\| = 1$ for all $x \in S$.

2.1. Inner product spaces and Hilbert spaces

Proposition 2.1.6 (*Pythagoras*). Assume $\{x_1, \dots, x_n\}$ is a finite orthogonal subset of an inner product space X . Then we have

$$\|x_1 + \dots + x_n\|^2 = \|x_1\|^2 + \dots + \|x_n\|^2.$$

Proposition 2.1.7. Assume $S = \{u_1, \dots, u_n\}$ is a finite orthonormal subset of an inner product space X . Then S is linearly independent. Moreover, if $u \in \text{Span}\{u_1, \dots, u_n\}$, i.e., if u is a linear combination of the vectors in S , then

$$u = \sum_{j=1}^n \langle u, u_j \rangle u_j \quad \text{and} \quad \|u\|^2 = \sum_{j=1}^n |\langle u, u_j \rangle|^2.$$

Proposition 2.1.8 (*Bessel's inequality*). Assume $S = \{u_j : j \in J\}$ is an orthonormal subset of an inner product space X indexed by a countable set J .¹ Then for any $x \in X$ we have

$$\sum_{j \in J} |\langle x, u_j \rangle|^2 \leq \|x\|^2.$$

Definition 2.1.9. An inner product space X (over $\mathbb{K}$) is called an *Hilbert space* (over $\mathbb{K}$) when X is complete as a normed space (with respect to the norm associated with its inner product).

Example 2.1.10. Since a finite-dimensional normed space is automatically complete we get that $\mathbb{K}^n$ is a Hilbert space w.r.t. its standard inner product. $\square$

Remark 2.1.11. Assume X is an inner product space. Considering X as a normed space, we may form its completion $\widetilde{X}$ (cf. Remark 1.1.6), and extend the inner product on X to an inner product on $\widetilde{X}$ as follows: if $y, y' \in \widetilde{X}$, then we can pick sequences $\{x_n\}_{n=1}^\infty, \{x'_n\}_{n=1}^\infty$ in X converging respectively to y and y' ; after checking that $\{\langle x_n, x'_n \rangle\}_{n=1}^\infty$ is a Cauchy sequence in $\mathbb{K}$, hence is convergent, we may set

$$\langle y, y' \rangle := \lim_{n \rightarrow \infty} \langle x_n, x'_n \rangle.$$

It is then a somewhat tedious exercise to verify that this gives a well-defined inner product on $\widetilde{X}$ which extends the one on X . This means that whenever needed, we may assume that X sits as a dense subspace of a Hilbert space $\widetilde{X}$ (called the *completion* of X) where the inner product on $\widetilde{X}$ extends the inner product on X . $\square$

¹We recall that a set J is *countable* if it is either finite or countably infinite (that is, there exists a bijection from $\mathbb{N}$ onto J).

2. On Hilbert spaces and bounded linear operators

We end this section with an important class of complex Hilbert spaces which is presumably well-known. (We could as well have considered real L^2 -spaces, but we will usually focus on the complex case.)

Example 2.1.12. Let $(X, \mathcal{A}, \mu)$ be a measure space and let $L^2 := L^2(X, \mathcal{A}, \mu)$ denote the associated complex L^2 -space. We use the convention that we identify measurable complex functions that agree almost everywhere on X w.r.t. μ , so we write elements of L^2 as functions. The Banach space $(L^2, \|\cdot\|_2)$ can be organized as a Hilbert space (over $\mathbb{C}$) as follows.

Let $f, g \in L^2$. Then $\bar{g} \in L^2$ and Hölder's inequality gives that $f\bar{g}$ is integrable w.r.t. μ . So we can set

$$\langle f, g \rangle := \int_X f \bar{g} \, d\mu.$$

It is an easy exercise to check that this gives an inner product on L^2 . As the associated norm obviously coincides with the $\|\cdot\|_2$ -norm, L^2 is complete w.r.t. this norm and we can conclude that L^2 is a Hilbert space. $\square$

2.2 Geometry in Hilbert spaces

We recall that if (X, d) is a metric space, $x \in X$ and A is a nonempty subset of X , then the *distance from x to A* is defined by

$$d(x, A) = \inf\{d(x, y) : y \in A\}.$$

If for example A is compact, then the function $y \mapsto d(x, y)$, being continuous, will attain its minimum on A ; hence, in this case, there exists some (not necessarily unique) $x_A \in A$ such that $d(x, A) = d(x, x_A)$. However, if A is closed, but not compact, such an element x_A may not exist (cf. Exercise 2.3).

Let us now consider a Hilbert space H with the metric d_H associated to its norm. If $x \in H$ and $M \neq \{0\}$ is a closed subspace of H , then M is not compact, so one can not use the result mentioned above. However, M is a convex² subset of H , and, using that the norm in a Hilbert space satisfies the parallelogram law, one can actually prove the following general result, valid for any closed convex subset of H .

² We recall that a subset C of a vector space V (over $\mathbb{K}$) is called *convex* if C contains the line segment between any two elements of C , i.e., if we have $(1-t)x + ty \in C$ whenever $x, y \in C$ and $t \in [0, 1]$.

2.2. Geometry in Hilbert spaces

Theorem 2.2.1. *Let C be a nonempty closed convex subset of a Hilbert space H and let $x \in H$. Then there is a unique vector $x_C \in C$ such that $d_H(x, x_C) = d_H(x, C)$, that is, such that*

$$\|x - x_C\| \leq \|x - y\| \quad \text{for all } y \in C.$$

The vector x_C is called the best approximation to x in C .

This result is proven in MAT3400/4400. Since a subspace is convex, Theorem 2.2.1 gives:

Corollary 2.2.2. *Let M be a closed subspace of a Hilbert space H and let $x \in H$. Then there is a unique vector $x_M \in M$, called the the best approximation to x in M , satisfying that*

$$\|x - x_M\| \leq \|x - y\| \quad \text{for all } y \in M.$$

This result has some far-reaching consequences. To help us formulate these, we first introduce some appropriate terminology.

Definition 2.2.3. Let M_1 and M_2 be subspaces of a vector space X over $\mathbb{K}$. We will say that X is the *algebraic direct sum* of M_1 and M_2 if every $x \in X$ can be written in a unique way as

$$x = m_1 + m_2 \quad \text{for some } m_1 \in M_1 \text{ and } m_2 \in M_2.$$

Moreover, if X is a normed space, then we will say that X is the *direct sum* of M_1 and M_2 , and write

$$X = M_1 \oplus M_2,$$

when M_1 and M_2 are both closed, and X is the algebraic direct sum of M_1 and M_2 .

In this course we will mainly be interested in the concept of direct sum in normed spaces. However, the following characterization of algebraic direct sum will be useful.

Lemma 2.2.4. *Let M_1 and M_2 be subspaces of a vector space X and set*

$$M_1 + M_2 := \{m_1 + m_2 : m_1 \in M_1, m_2 \in M_2\}.$$

Then the following two conditions are equivalent:

- (i) *X is the algebraic direct sum of M_1 and M_2 .*
- (ii) *$X = M_1 + M_2$ and $M_1 \cap M_2 = \{0\}$.*

2. On Hilbert spaces and bounded linear operators

Proof. Assume (ii) holds and let $x \in X$. Then we have $x = m_1 + m_2$ for some $m_1 \in M_1$ and $m_2 \in M_2$. Assume we also have $x = m'_1 + m'_2$ for some $m'_1 \in M_1$ and $m'_2 \in M_2$. Then we get

$$m_1 - m'_1 = m'_2 - m_2 \in M_1 \cap M_2.$$

Since $M_1 \cap M_2 = \{0\}$, this implies that $m'_1 = m_1$ and $m'_2 = m_2$. Thus (i) holds.

Conversely, assume (i) holds. It then obvious that $X = M_1 + M_2$. Consider $y \in M_1 \cap M_2$. Then we have $y = y + 0$ with $y \in M_1$, $0 \in M_2$, and $y = 0 + y$ with $0 \in M_1$, $y \in M_2$. As y has a unique decomposition as such a sum of vectors, we get that $y = 0$. This shows that $M_1 \cap M_2 = \{0\}$, hence that (i) holds. ■

Corollary 2.2.2 is the key to the following fundamental result.

Theorem 2.2.5. *Let M be a closed subspace of a Hilbert space H and set*

$$M^\perp = \{z \in H : \langle z, y \rangle = 0 \text{ for all } y \in M\}.$$

(i) *If $x \in H$ and $x_M \in M$ is the best approximation to x in M , then we have $x - x_M \in M^\perp$ and*

$$x = x_M + (x - x_M).$$

Moreover, $M^\perp$ is a closed subspace of H and we have

$$H = M \oplus M^\perp.$$

(ii) *The map $P_M : H \rightarrow H$ defined by $P_M(x) = x_M$ for $x \in H$ is called the orthogonal projection of H on M . We sometimes write Proj_M instead of P_M . It satisfies the following properties:*

- a) P_M is linear and bounded, with $\|P_M\| = 1$ if $M \neq \{0\}$.
- b) $(P_M)^2 = P_M$.
- c) $P_M(H) = M$ and $\ker(P_M) = M^\perp$.

(iii) *We also have*

$$(M^\perp)^\perp = M \quad \text{and} \quad P_{M^\perp} = I_H - P_M.$$

Proof. Part (i) is proven in MAT3400/4400.

(ii) We leave it as an exercise to verify that the map P_M is linear, and prove its other properties. Using Pythagoras' identity, we get that

$$\|P_M(x)\|^2 = \|x_M\|^2 \leq \|x_M\|^2 + \|x - x_M\|^2 = \|x_M + (x - x_M)\|^2 = \|x\|^2$$

for all $x \in H$. This shows that P_M is bounded with $\|P_M\| \leq 1$.

2.2. Geometry in Hilbert spaces

If $y \in M$, then $y = y + 0$ and $0 \in M^\perp$; this implies that $P_M(y) = y$. Hence, $\|P_M(y)\| = 1$ if $y \in M$ and $\|y\| = 1$. This implies that $\|P_M\| \geq 1$, hence that $\|P_M\| = 1$, if $M \neq \{0\}$.

Moreover, let $x \in H$. Since $P_M(x) = x_M \in M$ we get that

$$(P_M)^2(x) = P_M(x_M) = x_M = P_M(x).$$

Thus $(P_M)^2 = P_M$. We also see that $P_M(H) \subseteq M$. On the other hand, if $y \in M$, then $y = P_M(y) \in P_M(H)$. Thus $M \subseteq P_M(H)$, and it follows that $P_M(H) = M$.

Finally, assume that $x \in \ker(P_M)$, i.e., $x_M = 0$. Then

$$x = x - 0 = x - x_M \in M^\perp.$$

Conversely, assume that $x \in M^\perp$. Since $M^\perp$ is a subspace, we get that $x_M = x - (x - x_M) \in M^\perp$. Thus $x_M \in M \cap M^\perp = \{0\}$, i.e., $x_M = 0$. Altogether, this shows that $\ker(P_M) = M^\perp$.

(iii) Let $y \in M$. Then for all $z \in M^\perp$, we have $\langle y, z \rangle = \overline{\langle z, y \rangle} = 0$. This implies that $y \in (M^\perp)^\perp$. Hence we have $M \subseteq (M^\perp)^\perp$.

To show that $(M^\perp)^\perp \subseteq M$, we first observe that by applying the first part of the theorem to $M^\perp$, we get that

$$H = M^\perp \oplus (M^\perp)^\perp.$$

Now let $x \in (M^\perp)^\perp$, and set $x^\perp := x - x_M \in M^\perp$. Since $x_M \in M \subseteq (M^\perp)^\perp$, we can write

$$\begin{aligned} x &= x^\perp + x_M, \quad \text{where } x^\perp \in M^\perp \text{ and } x_M \in (M^\perp)^\perp, \text{ and} \\ x &= 0 + x, \quad \text{where } 0 \in M^\perp \text{ and } x \in (M^\perp)^\perp. \end{aligned}$$

By the uniqueness of decomposition in a direct sum, we get that $x = x_M$, so $x \in M$. Thus, we have shown that $(M^\perp)^\perp \subseteq M$, and we can conclude that $(M^\perp)^\perp = M$.

Finally, for $x \in H$, we have

$$x = (x - x_M) + x_M,$$

where $(x - x_M) \in M^\perp$ and $x_M \in M = (M^\perp)^\perp$. This gives that

$$P_{M^\perp}(x) = x - x_M = (I_H - P_M)(x).$$

Hence, $P_{M^\perp} = I_H - P_M$. ■

2. On Hilbert spaces and bounded linear operators

Remark 2.2.6. Assume that M is finite-dimensional subspace of a Hilbert space H and that $\mathcal{B} = \{u_1, \dots, u_n\}$ is an orthonormal basis for M . Then we know that the orthogonal projection P_M of H on M is given by

$$P_M(x) = \sum_{j=1}^n \langle x, u_j \rangle u_j \quad \text{for all } x \in H.$$

A similar formula holds when M is only assumed to be a closed subspace of H , as we will see in the next section after having discussed orthonormal bases in Hilbert spaces. $\square$

An immediate, but noteworthy, consequence of Theorem 2.2.5 is:

Corollary 2.2.7. *Let M be closed subspace of a Hilbert space H . Then $M = H$ if and only if $M^\perp = \{0\}$.*

In connection with the next corollary, we recall that if S is a nonempty subset of a vector space V , then $\text{Span}(S)$ denote the subspace of V consisting of all possible finite linear combinations of vectors in S .

Corollary 2.2.8. *Let S denote a nonempty subset of a Hilbert space H . Then $\text{Span}(S)$ is dense in H if and only if $S^\perp = \{0\}$.*

Proof. Set $M := \overline{\text{Span}(S)}$, which is a closed subspace of H . Then $\text{Span}(S)$ is dense in H if and only if $M = H$. As one easily verifies that $S^\perp = M^\perp$ (cf. Exercise 2.5), the result follows from Corollary 2.2.7. $\blacksquare$

A nonempty subset S of a normed space X is sometimes called *total in* X when $\text{Span}(S)$ is dense in X . So the corollary above says that S is total in H if and only if $S^\perp = \{0\}$.

Example 2.2.9. Let $(X, \mathcal{A}, \mu)$ be a measure space and $L^2 := L^2(X, \mathcal{A}, \mu)$.

Let $E \in \mathcal{A}$ and set $F := E^c \in \mathcal{A}$. If $g : X \rightarrow \mathbb{C}$ is measurable, we will say that g *lives essentially on* E when $\mu(\{x \in F : g(x) \neq 0\}) = 0$.

Note that if $h : X \rightarrow \mathbb{C}$ is measurable and $h = g$ μ -a.e., then h lives also essentially on E (you should check this). It makes therefore sense to define

$$M_E = \{g \in L^2 : g \text{ lives essentially on } E\}.$$

Similarly, we can define M_F . We claim that

$$M_F = (M_E)^\perp \quad \text{and} \quad M_E = (M_F)^\perp. \quad (2.2.1)$$

2.2. Geometry in Hilbert spaces

To prove this, assume first that $g \in M_E$ and $h \in M_F$. Then one easily checks that $g = g \mathbf{1}_E$ μ -a.e. and $h = h \mathbf{1}_F$ μ -a.e. As $E \cap F = \emptyset$, we get

$$\langle g, h \rangle = \int_X g \mathbf{1}_E \overline{h \mathbf{1}_F} \, d\mu = \int_X g \overline{h} \mathbf{1}_{E \cap F} \, d\mu = 0.$$

Since this is true for all $g \in M_E$, this implies that $h \in (M_E)^\perp$. As this holds for all $h \in M_F$, we get that $M_F \subseteq (M_E)^\perp$.

To show the reverse inclusion, let $h \in (M_E)^\perp$. Then we have

$$\int_X g \overline{h} \, d\mu = 0 \quad \text{whenever } g \in M_E.$$

In particular, since $h \mathbf{1}_E \in M_E$, we get

$$\int_X |h|^2 \mathbf{1}_E \, d\mu = \int_X (h \mathbf{1}_E) \overline{h} \, d\mu = 0.$$

Since $|h|^2 \mathbf{1}_E$ is nonnegative on X , this implies that

$$\mu(\{x \in X : |h(x)|^2 \mathbf{1}_E(x) \neq 0\}) = 0.$$

As $\{x \in E : h(x) \neq 0\} = \{x \in X : |h(x)|^2 \mathbf{1}_E(x) \neq 0\}$, we get that $\mu(\{x \in E : h(x) \neq 0\}) = 0$, hence that h lives essentially on F . Thus, $h \in M_F$. This shows that $(M_E)^\perp \subseteq M_F$.

Altogether, we have proved that $M_F = (M_E)^\perp$. Interchanging E and F , we get that $M_E = (M_F)^\perp$, and the proof of (2.2.1) is finished.

Since the orthogonal complement of any subset is a closed subspace, we can conclude from these equalities that M_E and M_F are closed subspaces of L^2 . Theorem 2.2.5 now gives that

$$L^2 = M_E \oplus (M_E)^\perp = M_E \oplus M_F.$$

Now, for every $f \in L^2$, we obviously have

$$f = f \mathbf{1}_E + f \mathbf{1}_F, \text{ where } f \mathbf{1}_E \in M_E \text{ and } f \mathbf{1}_F \in M_F = (M_E)^\perp.$$

Hence, Theorem 2.2.5 also gives that the orthogonal projection of L^2 on M_E (resp. M_F) is simply given by

$$P_{M_E}(f) = f \mathbf{1}_E \quad (\text{resp. } P_{M_F}(f) = f \mathbf{1}_F),$$

as one might have guessed. □

2.3 Orthonormal bases in Hilbert spaces

The notion of an orthonormal basis for a finite-dimensional inner product space, which is well-known from elementary linear algebra, have a natural generalization to Hilbert spaces.

Definition 2.3.1. A nonempty subset $\mathcal{B}$ of a Hilbert space H is called an *orthonormal basis* for H when $\mathcal{B}$ is orthonormal and $\text{Span}(\mathcal{B})$ is dense in H .

Suppose H is a (nonzero) finite-dimensional Hilbert space. Then an orthonormal set $\mathcal{B}$ in H has to be finite, so $\text{Span}(\mathcal{B})$, being finite-dimensional, is closed in H ; hence, $\text{Span}(\mathcal{B})$ is dense in H if and only if $\text{Span}(\mathcal{B}) = H$. Thus we see that Definition 2.3.1 agrees with the usual one when H is finite-dimensional. As a curiosity, we also mention that some authors like to define the empty set $\emptyset$ to be an orthonormal basis for the trivial Hilbert space $H = \{0\}$. As was shown in MAT 3400/4400, one can use Zorn's lemma to prove that every (non-zero) Hilbert space has an orthonormal basis.

The following example is of great importance in Fourier analysis (cf. MAT 3400/4400) and is worth being recalled.

Example 2.3.2. Let $H = L^2([-\pi, \pi], \mathcal{A}, \mu)$, where $\mathcal{A}$ denotes the σ -algebra of all Lebesgue measurable subsets of $[-\pi, \pi]$, and μ is the *normalized* Lebesgue measure on $\mathcal{A}$, that is,

$$\mu(A) := \frac{1}{2\pi} \lambda(A) \quad \text{for all } A \in \mathcal{A},$$

where λ denotes the Lebesgue measure on $\mathbb{R}$. In particular, we have $\mu([-\pi, \pi]) = 1$. For each $n \in \mathbb{Z}$, let $e_n : [-\pi, \pi] \rightarrow \mathbb{C}$ denote the continuous function given by

$$e_n(t) := e^{int} \quad \text{for all } t \in [-\pi, \pi].$$

Then the set $\mathcal{B} := \{e_n : n \in \mathbb{Z}\}$ is an orthonormal basis of H .

More generally, one may consider the L^2 -space associated to an interval $[a, b]$ and the normalized Lebesgue measure $\mu := \frac{1}{b-a} \lambda$. Then, letting e'_n be defined for each $n \in \mathbb{Z}$ by

$$e'_n(t) = e^{int \frac{2\pi}{b-a}} \quad \text{for all } t \in [a, b],$$

one may check that $\mathcal{B}' = \{e'_n : n \in \mathbb{Z}\}$ is an orthonormal basis. □

2.3. Orthonormal bases in Hilbert spaces

An immediate consequence of Corollary 2.2.8 is the following useful characterization of orthonormal bases:

Proposition 2.3.3. *Assume that $\mathcal{B}$ is an orthonormal subset of a Hilbert space H . Then $\mathcal{B}$ is an orthonormal basis for H if and only if $\mathcal{B}^\perp = \{0\}$.*

Example 2.3.4. Let X be a nonempty set. Then $\ell^2(X)$ has a natural orthonormal basis $\mathcal{B}$ which is the analogue of the standard basis $\{e_1, \dots, e_n\}$ for $\mathbb{K}^n$ (which one may identify with $\ell^2(\{1, \dots, n\})$).

Indeed, for each $x \in X$, let $e_x \in \ell^2(X)$ be defined by $e_x = \mathbf{1}_{\{x\}}$, and set

$$\mathcal{B} := \{e_x : x \in X\}.$$

Then $\mathcal{B}$ is clearly orthonormal. Moreover, let $f \in \ell^2(X)$, $f \in \mathcal{B}^\perp$. For each $x \in X$, we then have $\langle f, e_x \rangle = 0$. As

$$\langle f, e_x \rangle = \sum_{y \in X} f(y) \overline{e_x(y)} = \sum_{y \in \{x\}} f(y) = f(x),$$

we get that $f(x) = 0$ for all $x \in X$, i.e., $f = 0$. This shows that $\mathcal{B}^\perp = \{0\}$, and Proposition 2.3.3 gives that $\mathcal{B}$ is an orthonormal basis for $\ell^2(X)$. Note that $\mathcal{B}$ is uncountable if X is uncountable, e.g. if $X = \mathbb{R}$. $\square$

Example 2.3.5. The *Gram-Schmidt orthonormalization process*, of great usefulness in the finite-dimensional case, can be generalized to cover the following situation:

Let H be a Hilbert space, $H \neq \{0\}$. Let $\{x_j\}_{j \in \mathbb{N}}$ be a sequence of vectors in $H \setminus \{0\}$ and set $S := \{x_j : j \in \mathbb{N}\}$. Assume that $\text{Span}(S)$ is dense in H .

We remark that such a sequence exists whenever H is finite-dimensional (since repetitions are allowed in a sequence), and more generally whenever H is separable, i.e., contains a countable dense subset.

For each $n \in \mathbb{N}$, set $M_n := \text{Span}(\{x_1, \dots, x_n\})$. Clearly, for each $n \in \mathbb{N}$, we have $M_n \subseteq M_{n+1}$. Moreover, $\text{Span}(S) = \bigcup_{n \in \mathbb{N}} M_n$.

Proceeding inductively, we can construct an orthonormal basis $\mathcal{B}_n$ for each M_n as follows:

- i) We set $\mathcal{B}_1 := \left\{ \frac{1}{\|x_1\|} x_1 \right\}$. Clearly, $\mathcal{B}_1$ is an orthonormal basis for M_1 .
- ii) Let $n \in \mathbb{N}$ and assume that we have constructed an orthonormal basis $\mathcal{B}_n$ for M_n .

2. On Hilbert spaces and bounded linear operators

If $x_{n+1} \in M_n$, then set $\mathcal{B}_{n+1} := \mathcal{B}_n$. Otherwise, set

$$y_{n+1} := x_{n+1} - \text{Proj}_{M_n}(x_{n+1}) \quad \text{and} \quad \mathcal{B}_{n+1} := \mathcal{B}_n \cup \left\{ \frac{1}{\|y_{n+1}\|} y_{n+1} \right\}.$$

It follows readily that $\mathcal{B}_{n+1}$ is an orthonormal basis for M_{n+1} .

Set now $\mathcal{B} := \bigcup_{n \in \mathbb{N}} \mathcal{B}_n$. Then $\mathcal{B}$ is orthonormal, and $\text{Span}(\mathcal{B}) = \text{Span}(S)$, so

$$\overline{\text{Span}(\mathcal{B})} = \overline{\text{Span}(S)} = H.$$

Hence, $\mathcal{B}$ is an orthonormal basis for H . We note that $\mathcal{B}$ is countable since each $\mathcal{B}_n$ is finite. $\square$

When $\mathcal{B} = \{u_1, \dots, u_n\}$ is an orthonormal basis for a finite-dimensional Hilbert space H , we know that every $x \in H$ has a Fourier expansion w.r.t. $\mathcal{B}$, i.e., we have

$$x = \sum_{j=1}^n \langle x, u_j \rangle u_j.$$

As we will soon see, a similar expansion holds in any infinite dimensional Hilbert space.

We recall some notation. If J is a nonempty set (possibly uncountable), and $j \mapsto t_j$ is a function from J into $[0, \infty)$, we set

$$\sum_{j \in J} t_j := \sup \left\{ \sum_{j \in F} t_j : F \subseteq J, F \text{ is finite and nonempty} \right\} \in [0, \infty].$$

Equivalently, $\sum_{j \in J} t_j$ is the integral of the nonnegative function $j \mapsto t_j$ w.r.t. the counting measure on $\mathcal{P}(J)$ (= the σ -algebra of all subsets of J).

We first note that Bessel's inequality holds for any orthonormal set:

Lemma 2.3.6. *Assume that $\mathcal{B}$ is an orthonormal set in an inner product space H , and let $x \in H$. Then*

$$\sum_{u \in \mathcal{B}} |\langle x, u \rangle|^2 \leq \|x\|^2,$$

and the set $\mathcal{B}_x := \{u \in \mathcal{B} : \langle x, u \rangle \neq 0\}$ is countable.

Proof. Let F be a nonempty finite subset of $\mathcal{B}$. As F is orthonormal, Bessel's inequality for F gives that

$$\sum_{u \in F} |\langle x, u \rangle|^2 \leq \|x\|^2.$$

2.3. Orthonormal bases in Hilbert spaces

Thus we get that

$$\sup \left\{ \sum_{u \in F} |\langle x, u \rangle|^2 : F \subseteq \mathcal{B}, F \text{ is finite and nonempty} \right\} \leq \|x\|^2,$$

which proves the first assertion. Further, this implies that the set

$$\mathcal{B}_{x,n} := \{u \in \mathcal{B} : |\langle x, u \rangle|^2 \geq 1/n\}$$

is finite for every $n \in \mathbb{N}$. Hence, $\mathcal{B}_x = \bigcup_{n \in \mathbb{N}} \mathcal{B}_{x,n}$ is countable. ■

The next lemma will be useful at several occasions.

Lemma 2.3.7. *Assume $\{u_j : j \in \mathbb{N}\}$ is a countably infinite orthonormal subset of a Hilbert space H , and let $\{c_j\}_{j \in \mathbb{N}}$ be a sequence in $\mathbb{K}$ satisfying that*

$$\sum_{j=1}^{\infty} |c_j|^2 < \infty.$$

Then the series $\sum_{j=1}^{\infty} c_j u_j$ converges to some $y \in H$, and we have that

$$\langle y, u_k \rangle = c_k \text{ for every } k \in \mathbb{N}.$$

Proof. A similar result is shown in Lindstrøm's book, but we sketch the argument for the ease of the reader. For each $n \in \mathbb{N}$, set $y_n = \sum_{j=1}^n c_j u_j$. Then, for any $m > n$, Pythagoras' identity gives that

$$\|y_m - y_n\|^2 = \sum_{j=n+1}^m \|c_j u_j\|^2 = \sum_{j=n+1}^m |c_j|^2.$$

Using the assumption, the sum above can be made as small as we want by choosing m and n large enough. Thus the sequence $\{y_n\}_{n \in \mathbb{N}}$ is Cauchy in H , so it converges to some $y \in H$, i.e., we have $y = \sum_{j=1}^{\infty} c_j u_j$. For each $k \in \mathbb{N}$, continuity and linearity of the inner product in the first variable gives then that $\langle y, u_k \rangle = \sum_{j=1}^{\infty} c_j \langle u_j, u_k \rangle = c_k$. ■

Theorem 2.3.8. *Let H be a Hilbert space, $H \neq \{0\}$, and let $\mathcal{B}$ be an orthonormal subset of H . Then the following conditions are equivalent:*

- (a) $\mathcal{B}$ is an orthonormal basis for H .
- (b) Every $x \in H \setminus \{0\}$ has a Fourier expansion

$$x = \sum_{u \in \mathcal{B}_x} \langle x, u \rangle u \tag{2.3.1}$$

2. On Hilbert spaces and bounded linear operators

where $\mathcal{B}_x = \{u \in \mathcal{B} : \langle x, u \rangle \neq 0\}$ is countable (cf. Lemma 2.3.6) and nonempty.

When $\mathcal{B}_x$ is countably infinite, (2.3.1) means the following:

for any enumeration $\{u_j : j \in \mathbb{N}\}$ of the distinct elements of $\mathcal{B}_x$ we have

$$\lim_{n \rightarrow \infty} \left\| x - \sum_{j=1}^n \langle x, u_j \rangle u_j \right\| = 0, \text{ i.e., } x = \sum_{j=1}^{\infty} \langle x, u_j \rangle u_j.$$

(c) For every $x \in H$ we have $\|x\|^2 = \sum_{u \in \mathcal{B}} |\langle x, u \rangle|^2$.

The formula in (c) is called Parseval's identity.

Proof. (a) $\Rightarrow$ (b): Assume that $\mathcal{B}$ is an orthonormal basis for H and let $x \in H \setminus \{0\}$. From Lemma 2.3.6 we know that $\mathcal{B}_x$ is countable.

We first observe that $\mathcal{B}_x \neq \emptyset$. Indeed, suppose that $\mathcal{B}_x = \emptyset$. This means that $x \in \mathcal{B}^\perp$. But $\mathcal{B}^\perp = \{0\}$ by Proposition 2.3.3, so $x = 0$, a contradiction.

Next, we assume that $\mathcal{B}_x$ is countably infinite. (The case where $\mathcal{B}_x$ is finite is easier and left to the reader). Let then $\{u_j : j \in \mathbb{N}\}$ be an enumeration of the distinct elements of $\mathcal{B}_x$. Since $\mathcal{B}_x$ is orthonormal, Bessel's inequality gives that

$$\sum_{j=1}^{\infty} |\langle x, u_j \rangle|^2 \leq \|x\|^2.$$

Applying Lemma 2.3.7 with $c_j = \langle x, u_j \rangle$ for every $j \in \mathbb{N}$, we get that the series $\sum_{j=1}^{\infty} \langle x, u_j \rangle u_j$ converges to some $y \in H$, which satisfies that

$$\langle y, u_k \rangle = c_k = \langle x, u_k \rangle \quad \text{for every } k \in \mathbb{N}.$$

Moreover, if $u \in \mathcal{B} \setminus \mathcal{B}_x$, we get that

$$\langle y, u \rangle = \sum_{j=1}^{\infty} \langle x, u_j \rangle \langle u_j, u \rangle = 0 = \langle x, u \rangle.$$

It follows that $x - y \in \mathcal{B}^\perp = \{0\}$, hence that $x = y$. This shows that the assertion in (b) holds in this case.

(b) $\Rightarrow$ (c): Assume (b) holds, and let $x \in H \setminus \{0\}$. Again we consider the more difficult case where $\mathcal{B}_x$ is countably infinite, so we may write $\mathcal{B}_x = \{u_j : j \in \mathbb{N}\}$ as above. By continuity of the norm and Pythagoras' identity, we get

$$\|x\|^2 = \sum_{j=1}^{\infty} |\langle x, u_j \rangle|^2.$$

2.3. Orthonormal bases in Hilbert spaces

Hence, given $\varepsilon > 0$, we can find $n \in \mathbb{N}$ such that $\|x\|^2 - \sum_{j=1}^n |\langle x, u_j \rangle|^2 < \varepsilon$, giving

$$\|x\|^2 - \varepsilon < \sum_{j=1}^n |\langle x, u_j \rangle|^2 \leq \sum_{u \in \mathcal{B}} |\langle x, u \rangle|^2.$$

Since this holds for every $\varepsilon > 0$, we get that $\|x\|^2 \leq \sum_{u \in \mathcal{B}} |\langle x, u \rangle|^2$. Combining this inequality with Lemma 2.3.6, we see that (c) holds.

(c) $\Rightarrow$ (a): Assume $\|x\|^2 = \sum_{u \in \mathcal{B}} |\langle x, u \rangle|^2$ for every $x \in H$. If $x \in \mathcal{B}^\perp$, i.e., $\langle x, u \rangle = 0$ for every $u \in \mathcal{B}$, then we get $\|x\|^2 = 0$, so $x = 0$. Hence, $\mathcal{B}^\perp = \{0\}$, and Proposition 2.3.3 gives that (a) holds. ■

Remark 2.3.9. The Fourier expansion of x in Theorem 2.3.8 (b) can be written in the form

$$x = \sum_{u \in \mathcal{B}} \langle x, u \rangle u \quad (2.3.2)$$

if one takes care of giving a meaning to convergence of *generalized sums* in normed spaces. We discuss this in Exercise 2.15. In these notes, we will sometimes use (2.3.2) as a short form of the Fourier expansion of x given by (2.3.1). □

Example 2.3.10. Let M be a closed subspace of a Hilbert space H and suppose that we have found an orthonormal basis $\mathcal{C}$ for M . Then we can find a formula for the orthogonal projection P_M of H on M :

Let $x \in H$. If $x \in M^\perp$, then $P_M(x) = 0$, so we can assume $x \in H \setminus M^\perp$. Since $\mathcal{C}$ is orthonormal in H , we know that $\mathcal{C}_x := \{v \in \mathcal{C} : \langle x, v \rangle \neq 0\}$ is countable.

Set $x_M := P_M(x) \in M$ and $x^\perp := x - x_M \in M^\perp$, and note that $x_M \neq 0$. Now, for each $v \in \mathcal{C}$, we have

$$\langle x, v \rangle = \langle x_M, v \rangle + \langle x^\perp, v \rangle = \langle x_M, v \rangle.$$

Hence, $\mathcal{C}_x = \mathcal{C}_{x_M}$. Moreover, applying Theorem 2.3.8 to M , $x_M \in M \setminus \{0\}$ and $\mathcal{C}$, we get that

$$x_M = \sum_{v \in \mathcal{C}_{x_M}} \langle x_M, v \rangle v.$$

Using our previous observations, this formula can be rewritten as

$$P_M(x) = \sum_{v \in \mathcal{C}_x} \langle x, v \rangle v.$$

It generalizes the usual formula for $P_M(x)$ when M is finite-dimensional. □

2. On Hilbert spaces and bounded linear operators

An immediate consequence of Theorem 2.3.8 is the following:

Corollary 2.3.11. *Assume a Hilbert space H contains a countably infinite orthonormal subset $\mathcal{B}$, enumerated as $\mathcal{B} = \{v_k : k \in \mathbb{N}\}$. Then $\mathcal{B}$ is an orthonormal basis for H if and only if*

$$x = \sum_{k=1}^{\infty} \langle x, v_k \rangle v_k$$

for all $x \in H$, if and only if

$$\|x\|^2 = \sum_{k=1}^{\infty} |\langle x, v_k \rangle|^2$$

for all $x \in H$.

Example 2.3.12. Let $\mathcal{B} = \{e_n : n \in \mathbb{Z}\}$ denote the standard orthonormal basis for $H = L^2([-\pi, \pi])$ described in Example 2.3.2. For $f \in H$ and $n \in \mathbb{Z}$ it is common to set

$$\widehat{f}(n) := \langle f, e_n \rangle = \frac{1}{2\pi} \int_{[-\pi, \pi]} f(t) e^{-int} d\lambda(t),$$

which is called the *Fourier coefficient* of f at n . The Fourier expansion of f w.r.t. $\mathcal{B}$ is then written as

$$f = \sum_{n \in \mathbb{Z}} \widehat{f}(n) e_n,$$

meaning that

$$f = \lim_{m \rightarrow \infty} \sum_{n=-m}^m \widehat{f}(n) e_n \quad (\text{w.r.t. } \|\cdot\|_2).$$

This follows from Corollary 2.3.11 by enumerating $\mathcal{B}$ as $e_0, e_{-1}, e_1, e_{-2}, e_2, \dots$ etc. Similarly, we have

$$\|f\|_2 = \left(\sum_{n \in \mathbb{Z}} |\widehat{f}(n)|^2 \right)^{1/2}. \quad \square$$

2.4 Adjoint operators

Let X be a normed space (over $\mathbb{K}$). We recall that the dual space X^* consists of the bounded linear functionals on X (with values in $\mathbb{K}$), and that X^* is a Banach space w.r.t. the norm $\|\varphi\| := \sup\{|\varphi(x)| : x \in X_1\}$, $\varphi \in X^*$. An important goal of functional analysis is to gain new insight by exploiting the interplay between a space and its dual. This is particularly successful when X is a Hilbert space because the dual space may then be identified in a natural way with the space itself, as was shown in MAT3400/4400.

Theorem 2.4.1. *Let H be a Hilbert space (over $\mathbb{K}$). For each $y \in H$, define $\varphi_y : H \rightarrow \mathbb{K}$ by*

$$\varphi_y(x) := \langle x, y \rangle \quad \text{for all } x \in H.$$

Then $\varphi_y \in H^*$ for all $y \in H$.

Moreover, the map $y \mapsto \varphi_y$ is a bijection from H onto H^* , which is isometric, and conjugate-linear in the sense that

$$\varphi_{\lambda_1 y_1 + \lambda_2 y_2} = \overline{\lambda_1} \varphi_{y_1} + \overline{\lambda_2} \varphi_{y_2}$$

for all $\lambda_1, \lambda_2 \in \mathbb{K}$ and all $y_1, y_2 \in H$.

This theorem, which is one among a diversity of results being called *the Riesz representation theorem*, has several useful consequences. Our main application in these notes will be to use it to associate an adjoint operator to every bounded operator on a Hilbert space. Some people like to think of the adjoint as a kind of twin (or as a kind of shadow), which happens to coincide with the original operator in many cases of interest.

Theorem 2.4.2. *Let H be a Hilbert space (over $\mathbb{K}$). For each $T \in \mathcal{B}(H)$, there is a unique operator $T^* \in \mathcal{B}(H)$, called the adjoint of T , satisfying*

$$\langle T(x), y \rangle = \langle x, T^*(y) \rangle \quad (2.4.1)$$

for all $x, y \in H$.

The $*$ -operation on $\mathcal{B}(H)$, $T \mapsto T^*$, enjoys the following properties:

For all $S, T \in \mathcal{B}(H)$ and all $\alpha, \beta \in \mathbb{K}$, we have

- *i) $(\alpha S + \beta T)^* = \overline{\alpha} S^* + \overline{\beta} T^*$; ii) $(ST)^* = T^* S^*$; iii) $(T^*)^* = T$;*
- *iv) $\|T^*\| = \|T\|$; v) $\|T^* T\| = \|T\|^2$.*

Remark 2.4.3. If H and K are Hilbert spaces (over the same $\mathbb{K}$), then one may associate to each $T \in \mathcal{B}(H, K)$ a unique adjoint operator $T^* \in \mathcal{B}(K, H)$ satisfying (2.4.1) for all $x \in H$ and all $y \in K$, and enjoying similar properties. We leave this as an exercise. $\square$

Proof of Theorem 2.4.2. Let $T \in \mathcal{B}(H)$ and consider $y \in H$. Using the linearity of T and the linearity of the inner product in the first variable, we get that the map $\varphi : H \rightarrow \mathbb{K}$ defined by

$$\varphi(x) := \langle T(x), y \rangle \quad \text{for all } x \in H,$$

is a linear functional on H . Moreover, as we have

$$|\langle T(x), y \rangle| \leq \|T(x)\| \|y\| \leq \|T\| \|x\| \|y\|$$

2. On Hilbert spaces and bounded linear operators

for all $x \in H$, φ is bounded with $\|\varphi\| \leq \|T\| \|y\|$. Hence, $\varphi \in H^*$, and Theorem 2.4.1 gives that there exists a unique vector in H , that we denote by $T^*(y)$, such that $\varphi = \varphi_{T^*(y)}$, i.e., such that

$$\langle T(x), y \rangle = \langle x, T^*(y) \rangle \quad (2.4.2)$$

for all $x \in H$. This theorem also gives that

$$\|T^*(y)\| = \|\varphi_{T^*(y)}\| = \|\varphi\| \leq \|T\| \|y\|. \quad (2.4.3)$$

Since what we have done above holds for every $y \in H$, we obtain a map $T^* : H \rightarrow H$ which sends each $y \in H$ to $T^*(y) \in H$. In view of (2.4.2), it is clear that (2.4.1) holds for all $x, y \in H$.

To show that T^* is linear, let $y, y' \in H$ and $\alpha \in \mathbb{K}$. Then, for all $x \in H$, we have

$$\begin{aligned} \langle x, T^*(\alpha y + y') \rangle &= \langle T(x), \alpha y + y' \rangle \\ &= \overline{\alpha} \langle T(x), y \rangle + \langle T(x), y' \rangle \\ &= \overline{\alpha} \langle x, T^*(y) \rangle + \langle x, T^*(y') \rangle \\ &= \langle x, \alpha T^*(y) + T^*(y') \rangle. \end{aligned}$$

This implies that $T^*(\alpha y + y') = \alpha T^*(y) + T^*(y')$, as desired.

Next, from (2.4.3), we see that T^* is bounded with $\|T^*\| \leq \|T\|$. To show the asserted uniqueness property of T^* , assume that $S \in \mathcal{B}(H)$ satisfies the same property as T^* , i.e.,

$$\langle T(x), y \rangle = \langle x, S(y) \rangle \quad \text{for all } x, y \in H.$$

Let $y \in H$. Then, for all $x \in H$, we get

$$\langle x, S(y) \rangle = \langle T(x), y \rangle = \langle x, T^*(y) \rangle.$$

This implies that $S(y) = T^*(y)$. Thus, $S = T^*$.

We leave the proof of properties *i*) and *ii*) as an exercise. To show the other properties, let $T \in \mathcal{B}(H)$. Then, for each $y \in H$, using equation (2.4.1) for T^* instead of T , we get that, for all $x \in H$, we have

$$\begin{aligned} \langle x, (T^*)^*(y) \rangle &= \langle T^*(x), y \rangle = \overline{\langle y, T^*(x) \rangle} \\ &= \overline{\langle T(y), x \rangle} = \langle x, T(y) \rangle \end{aligned}$$

This implies that $(T^*)^*(y) = T(y)$. Thus, $(T^*)^* = T$, i.e., *iii*) holds

Now, we have seen that $\|T^*\| \leq \|T\|$ holds for all $T \in \mathcal{B}(H)$. Thus we get

$$\|T\| = \|(T^*)^*\| \leq \|T^*\| \leq \|T\|.$$

Hence $\|T^*\| = \|T\|$, i.e., *iv*) holds.

Further, using *iv*), we get $\|T^*T\| \leq \|T^*\|\|T\| = \|T\|^2$. On the other hand, for every $x \in H$, we have

$$\begin{aligned} \|T(x)\|^2 &= \langle T(x), T(x) \rangle = \langle x, T^*(T(x)) \rangle \\ &= |\langle x, (T^*T)(x) \rangle| \leq \|x\| \|(T^*T)(x)\| \\ &\leq \|T^*T\| \|x\|^2. \end{aligned}$$

This implies that $\|T\|^2 \leq \|T^*T\|$. Hence we get $\|T\|^2 = \|T^*T\|$, i.e., *v*) holds. ■

Example 2.4.4. Consider $H = \mathbb{K}^n$ for some $n \in \mathbb{N}$ with its usual inner product, and $T \in \mathcal{B}(H)$. Let A denote the standard matrix of T . Then the standard matrix of T^* is $A^* := \overline{A}^t$.

Here, $\overline{A}$ denotes the matrix obtained by conjugating every coefficient of A , while B^t denotes the transpose of a matrix B . Of course, if $\mathbb{K} = \mathbb{R}$, then we just get $A^* = A^t$.

We can reformulate our assertion above by saying that if $T_A \in \mathcal{B}(H)$ denotes the operator given by multiplication with a matrix $A \in M_n(\mathbb{K})$, then we have

$$(T_A)^* = T_{A^*}.$$

To prove this, let $x, y \in H$. Recall that $\langle x, y \rangle = x^t \overline{y}$. So we get

$$\langle T_A(x), y \rangle = (Ax)^t \overline{y} = x^t A^t \overline{y} = x^t \overline{A^* y} = \langle x, T_{A^*}(y) \rangle.$$

Since this holds for all $x, y \in H$, this implies that $(T_A)^* = T_{A^*}$, as asserted.

More generally, if H is a nontrivial finite-dimensional Hilbert space, $\mathcal{B}$ is an orthonormal basis for H , and $[T]_{\mathcal{B}}$ is the matrix of T relative to $\mathcal{B}$, then we have

$$[T^*]_{\mathcal{B}} = ([T]_{\mathcal{B}})^*.$$

We leave the verification of this fact is left as an exercise. □

Example 2.4.5. Assume H is a Hilbert space having a countably infinite orthonormal basis $\mathcal{B}$, enumerated as $\mathcal{B} = \{u_j\}_{j \in \mathbb{N}}$. Let $\{\lambda_j\}_{j \in \mathbb{N}}$ be a bounded sequence in $\mathbb{K}$, and set

$$M := \sup\{|\lambda_j| : j \in \mathbb{N}\} < \infty.$$

2. On Hilbert spaces and bounded linear operators

We claim that there exists an operator $D \in \mathcal{B}(H)$ satisfying

$$D(u_k) = \lambda_k u_k \quad \text{for each } k \in \mathbb{N}. \quad (2.4.4)$$

Indeed, consider $x \in H$. Then Parseval's identity gives that

$$\sum_{j=1}^{\infty} |\lambda_j \langle x, u_j \rangle|^2 \leq M^2 \sum_{j=1}^{\infty} |\langle x, u_j \rangle|^2 = M^2 \|x\|^2 < \infty.$$

Hence, Lemma 2.3.7 gives that the vector

$$D(x) := \sum_{j=1}^{\infty} \lambda_j \langle x, u_j \rangle u_j$$

satisfies that $\langle D(x), u_j \rangle = \lambda_j \langle x, u_j \rangle$ for each $j \in \mathbb{N}$. Thus, using Parseval's identity again, we get that

$$\|D(x)\|^2 = \sum_{j=1}^{\infty} |\lambda_j \langle x, u_j \rangle|^2 \leq M^2 \|x\|^2.$$

It follows now readily that the map $x \mapsto D(x)$ gives an operator D in $\mathcal{B}(H)$ satisfying (2.4.4) and $\|D\| \leq M$. Since

$$\|D\| \geq \|D(u_k)\| = \|\lambda_k u_k\| = |\lambda_k|$$

for all $k \in \mathbb{N}$, we also have that $\|D\| \geq M$. Hence, $\|D\| = M$.

The operator D is called *the diagonal operator associated to $\{\lambda_j\}_{j \in \mathbb{N}}$ (w.r.t. $\mathcal{B}$)*.

We note that the adjoint of D is the diagonal operator associated to $\{\overline{\lambda_k}\}_{k \in \mathbb{N}}$ (w.r.t. $\mathcal{B}$), that is, D^* satisfies that $D^*(u_k) = \overline{\lambda_k} u_k$ for all $k \in \mathbb{N}$.

Indeed, let $k \in \mathbb{N}$. Then, for any $x \in H$, we have:

$$\begin{aligned} \langle x, D^*(u_k) \rangle &= \langle D(x), u_k \rangle = \left\langle \sum_{j=1}^{\infty} \lambda_j \langle x, u_j \rangle u_j, u_k \right\rangle \\ &= \sum_{j=1}^{\infty} \lambda_j \langle x, u_j \rangle \langle u_j, u_k \rangle = \lambda_k \langle x, u_k \rangle = \langle x, \overline{\lambda_k} u_k \rangle. \end{aligned}$$

Now, if $y, z \in H$, we have $y = z \Leftrightarrow \langle x, y \rangle = \langle x, z \rangle$ for all $x \in H$. (Why? Think of it!) Thus this implies that $D^*(u_k) = \overline{\lambda_k} u_k$, as desired.

Note that if $H = \ell^2$ and $\mathcal{B} = \{e_j\}_{j \in \mathbb{N}}$ is the standard orthonormal basis of H , then the diagonal operator associated to a bounded sequence $\{\lambda_j\}_{j \in \mathbb{N}}$ (w.r.t. $\mathcal{B}$) is the operator $D : \ell^2 \rightarrow \ell^2$ given by

$$[D(\xi)](n) = \lambda_n \xi(n) \quad \text{for all } n \in \mathbb{N}.$$

□

Example 2.4.6. Let again H be a Hilbert space having a countably infinite orthonormal basis, enumerated as $\mathcal{B} = \{u_j\}_{j \in \mathbb{N}}$. Another interesting operator in $\mathcal{B}(H)$ is *the right shift operator S on H (w.r.t. $\mathcal{B}$)*, which satisfies

$$S(u_k) = u_{k+1} \quad \text{for all } k \in \mathbb{N}, \quad (2.4.5)$$

i.e., $S(u_1) = u_2$, $S(u_2) = u_3$, etc. We first show that S exists. Since

$$\sum_{n=2}^{\infty} |\langle x, u_{n-1} \rangle|^2 = \sum_{j=1}^{\infty} |\langle x, u_j \rangle|^2 = \|x\|^2 < \infty$$

for all $x \in H$, we may use Lemma 2.3.7 to define a map $S : H \rightarrow H$ by

$$S(x) = \sum_{n=2}^{\infty} \langle x, u_{n-1} \rangle u_n.$$

It is easy to verify that S is a linear isometry satisfying that $S(u_{n-1}) = u_n$ for all $n \geq 2$, i.e., such that (2.4.5) holds.

The adjoint of S is the bounded operator S^* on H which is determined by

$$S^*(u_k) = \begin{cases} 0 & \text{if } k = 1, \\ u_{k-1} & \text{if } k \geq 2 \end{cases}$$

for $k \in \mathbb{N}$. Indeed, for any $x \in H$, we have

$$\langle x, S^*(u_1) \rangle = \langle S(x), u_1 \rangle = \sum_{n=2}^{\infty} \langle x, u_{n-1} \rangle \langle u_n, u_1 \rangle = 0 = \langle x, 0 \rangle,$$

while for $k \in \mathbb{N}$, $k \geq 2$, we get

$$\langle x, S^*(u_k) \rangle = \langle S(x), u_k \rangle = \sum_{n=2}^{\infty} \langle x, u_{n-1} \rangle \langle u_n, u_k \rangle = \langle x, u_{k-1} \rangle.$$

The operator S^* is called *the left shift operator on H (w.r.t. $\mathcal{B}$)*. We note that S^* is not isometric; in fact, it is not even injective since $S^*(u_1) = 0$. $\square$

Example 2.4.7. Let $(X, \mathcal{A}, \mu)$ be a measure space. Set $L^\infty := L^\infty(X, \mathcal{A}, \mu)$ and $H := L^2(X, \mathcal{A}, \mu)$. To each $f \in L^\infty$ we may associate a "multiplication" operator $M_f \in \mathcal{B}(H)$ given by

$$M_f(g) = fg \quad \text{for all } g \in H.$$

(You should check that this is meaningful.) Since for all $g, h \in H$ we have

$$\langle M_f(g), h \rangle = \int_X fg \bar{h} \, d\mu = \int_X g \overline{f h} \, d\mu = \langle g, M_{\bar{f}}(h) \rangle,$$

we get that $(M_f)^* = M_{\bar{f}}$. $\square$

2. On Hilbert spaces and bounded linear operators

Example 2.4.8. Let $K : [a, b] \times [a, b] \rightarrow \mathbb{C}$ be a continuous function and let T_K denote the associated integral operator on $H = L^2([a, b])$ determined on $C([a, b])$ by

$$[T_K(f)](s) = \int_a^b K(s, t) f(t) dt \quad \text{for all } s \in [a, b],$$

cf. Example 1.3.6. We leave it as Exercise 2.18 to check that $(T_K)^* = T_{K^*}$, where $K^*(s, t) := \overline{K(t, s)}$ for all $s, t \in [a, b]$. $\square$

Example 2.4.9. Let $v, w \in H$ and consider the linear operator $T_{v,w} : H \rightarrow H$ defined by

$$T_{v,w}(x) := \langle x, v \rangle w \quad \text{for all } x \in H.$$

Note that if $v, w \in H \setminus \{0\}$, then $T_{v,w}$ has rank one, i.e., its range has dimension one. It is an instructive exercise (cf. Exercise 2.20) to check that $T_{v,w}$ is bounded, with norm $\|T_{v,w}\| = \|v\| \|w\|$, and satisfies $(T_{v,w})^* = T_{w,v}$. One may also show that if $T \in \mathcal{B}(H)$ has finite-rank, meaning that the range of T is finite-dimensional, then T may be written as a finite sum of rank one operators in $\mathcal{B}(H)$, which implies that T^* has finite-rank too. $\square$

As an illustration that the adjoint operator contains valuable information about the original operator, we include a proposition showing the connection between the fundamental subspaces associated to these operators.

Proposition 2.4.10. *Let H be a Hilbert space (over $\mathbb{K}$) and let $T \in \mathcal{B}(H)$. Then we have:*

- (a) $\ker(T) = T^*(H)^\perp$ and $\ker(T^*) = T(H)^\perp$.
- (b) $\overline{T(H)} = \ker(T^*)^\perp$ and $\overline{T^*(H)} = \ker(T)^\perp$.

Proof. Both equalities in (a) are immediate consequences of (2.4.1). Using Exercise 2.5 with $N = T(H)$, we get that $\overline{T(H)} = (T(H)^\perp)^\perp = \ker(T^*)^\perp$. The second equality in (b) is shown similarly (or by replacing T with T^* in the first one). $\blacksquare$

Corollary 2.4.11. *Let H be a Hilbert space (over $\mathbb{K}$) and let $T \in \mathcal{B}(H)$. Then $T(H)$ is dense in H if and only if T^* is 1-1.*

Proof. Using Proposition 2.4.10 and Corollary 2.2.7, we get

$$\overline{T(H)} = H \Leftrightarrow \ker(T^*)^\perp = H \Leftrightarrow \ker(T^*) = \{0\}.$$

As T^* is linear, we also have $\ker(T^*) = \{0\} \Leftrightarrow T^*$ is 1-1. $\blacksquare$

As another illustration, we finally mention:

Proposition 2.4.12. *Let H be a Hilbert space (over $\mathbb{K}$) and let $T \in \mathcal{B}(H)$. Then T is invertible in $\mathcal{B}(H)$ if and only if T^* is invertible in $\mathcal{B}(H)$, in which case we have $(T^*)^{-1} = (T^{-1})^*$.*

Proof. Left to the reader as Exercise 2.19. ■

2.5 Self-adjoint operators

In this section, we introduce a very important class of bounded operators on a Hilbert space and discuss some of their properties.

Definition 2.5.1. Let H be a Hilbert space (over $\mathbb{K}$). An operator $T \in \mathcal{B}(H)$ is called *self-adjoint* when $T^* = T$, that is, we have

$$\langle T(x), y \rangle = \langle x, T(y) \rangle \quad \text{for all } x, y \in H.$$

If $\mathbb{K} = \mathbb{C}$, a self-adjoint operator in $\mathcal{B}(H)$ is also called *Hermitian*, while it is often called *symmetric* if $\mathbb{K} = \mathbb{R}$.

We note that if $T, T' \in \mathcal{B}(H)$ are self-adjoint, and $\lambda \in \mathbb{R}$, then it is obvious that $\lambda T + T'$ is also self-adjoint.

Example 2.5.2. Let $A = [a_{j,k}] \in M_n(\mathbb{K})$ and let $T_A \in \mathcal{B}(\mathbb{K}^n)$ denote the operator given by multiplication with A (cf. Example 2.4.4). Then T_A is self-adjoint if and only if $A^* = A$, i.e., $\overline{a_{k,j}} = a_{j,k}$ for all $j, k \in \{1, \dots, n\}$. In particular, when $\mathbb{K} = \mathbb{R}$, T_A is self-adjoint if and only if A is symmetric. □

Example 2.5.3. Assume H is a Hilbert space with a countably infinite orthonormal basis $\mathcal{B}$, enumerated as $\mathcal{B} = \{u_j\}_{j \in \mathbb{N}}$.

Let D denote the diagonal operator associated to a bounded sequence $\{\lambda_j\}_{j \in \mathbb{N}}$ in $\mathbb{K}$ (w.r.t. $\mathcal{B}$), so $D(u_j) = \lambda_j u_j$ for every j , cf. Example 2.4.5. Then D is self-adjoint if and only if $\overline{\lambda_j} = \lambda_j$ for all $j \in \mathbb{N}$, i.e., $\lambda_j \in \mathbb{R}$ for all $j \in \mathbb{N}$. In particular, D is always self-adjoint when $\mathbb{K} = \mathbb{R}$.

Let S denote the right shift operator on H (w.r.t. $\mathcal{B}$), so $S(u_j) = u_{j+1}$ for every $j \in \mathbb{N}$. As we have seen in Example 2.4.6, S^* is the left shift operator, and it is obvious that $S^* \neq S$. So S is not self-adjoint. □

Example 2.5.4. Let $(X, \mathcal{A}, \mu)$ be a measure space and set $H := L^2(X, \mathcal{A}, \mu)$. Let $f \in L^\infty(X, \mathcal{A}, \mu)$ and consider the multiplication operator $M_f \in \mathcal{B}(H)$, cf. Example 2.4.7. Then M_f is self-adjoint if and only if $M_{\bar{f}} = M_f$.

Thus, M_f is self-adjoint whenever f is real-valued (μ -a.e.). It can be shown that the converse statement holds whenever $(X, \mathcal{A}, \mu)$ is assumed to be semifinite (cf. Exercise 2.24).

2. On Hilbert spaces and bounded linear operators

Example 2.5.5. Let $K : [a, b] \times [a, b] \rightarrow \mathbb{C}$ be a continuous function and let T_K denote the associated integral operator on $H = L^2([a, b])$, cf. Example 2.4.8. Then, since $(T_K)^* = T_{K^*}$ (where $K^*(s, t) = \overline{K(t, s)}$), we get that T_K is self-adjoint if and only if $T_{K^*} = T_K$. Hence it is clear that T_K is self-adjoint whenever K is real-valued. We leave it as an exercise to check that the converse statement also holds. $\square$

Example 2.5.6. Let M be a closed subspace of a Hilbert space H and let P_M denote the orthogonal projection of H on M . Then P_M is self-adjoint. Indeed, let $x, y \in H$. As $P_M(x) \in M$ and $y - P_M(y) \in M^\perp$, we have

$$\langle P_M(x), y - P_M(y) \rangle = 0.$$

Similarly, we have $\langle x - P_M(x), P_M(y) \rangle = 0$. Hence we get

$$\begin{aligned} \langle P_M(x), y \rangle &= \langle P_M(x), P_M(y) + y - P_M(y) \rangle \\ &= \langle P_M(x), P_M(y) \rangle + \langle P_M(x), y - P_M(y) \rangle \\ &= \langle P_M(x), P_M(y) \rangle \\ &= \langle P_M(x), P_M(y) \rangle + \langle x - P_M(x), P_M(y) \rangle \\ &= \langle P_M(x) + x - P_M(x), P_M(y) \rangle \\ &= \langle x, P_M(y) \rangle \end{aligned}$$

$\square$

An orthogonal projection map P_M has another property, namely that it is a *positive* operator on H . Such operators are generalizations of operators associated with positive semidefinite matrices. We refer to Exercise 2.29 for the definition and some of the properties of positive operators.

Next, we note that there is an abundance of self-adjoint operators.

Proposition 2.5.7. *Let H be a Hilbert space (over $\mathbb{K}$) and $T \in \mathcal{B}(H)$.*

Then the operators $T + T^*$, T^*T and TT^* are all self-adjoint. Moreover, if $\mathbb{K} = \mathbb{C}$, then $\frac{1}{i}(T - T^*)$ is also self-adjoint.

Proof. The reader should have no difficulty to verify these assertions by using the properties of the $*$ -operation on $\mathcal{B}(H)$ listed in Theorem 2.4.2. $\blacksquare$

A noteworthy consequence is that bounded self-adjoint operators on a *complex* Hilbert space have a canonical decomposition similar to the one enjoyed by complex numbers.

2.5. Self-adjoint operators

Corollary 2.5.8. *Let H be a Hilbert space over $\mathbb{C}$ and let $T \in \mathcal{B}(H)$. Set*

$$\operatorname{Re}(T) := \frac{1}{2}(T + T^*), \quad \operatorname{Im}(T) := \frac{1}{2i}(T - T^*).$$

Then $\operatorname{Re}(T)$ and $\operatorname{Im}(T)$ are both self-adjoint, and we have

$$T = \operatorname{Re}(T) + i \operatorname{Im}(T).$$

Proof. The first assertion follows readily from Proposition 2.5.7. The second one is elementary. ■

Consider a bounded operator T on a Hilbert space $H \neq \{0\}$. The *numerical range* of T is defined as the subset of $\mathbb{K}$ given by

$$W_T := \left\{ \langle T(x), x \rangle : x \in H, \|x\| = 1 \right\}.$$

Some properties of T are reflected in the geometric properties of W_T and $\overline{W_T}$, cf. Exercise 2.34. We will mainly be interested in the *numerical radius* of T , which is defined by

$$N_T := \sup\{|\lambda| : \lambda \in W_T\} = \sup\{|\langle T(x), x \rangle| : x \in H, \|x\| = 1\}.$$

We note that the Cauchy-Schwarz inequality implies that $N_T \leq \|T\|$.

A remarkable fact, proven below, is that N_T agrees with $\|T\|$ when T is self-adjoint. We first observe that if T is self-adjoint, then $W_T \subseteq \mathbb{R}$. Indeed, if $T^* = T$, then for every $x \in H$, we have

$$\langle T(x), x \rangle = \langle x, T(x) \rangle = \overline{\langle T(x), x \rangle},$$

and the claim clearly follows.

Theorem 2.5.9. *Let H be a Hilbert space, $H \neq \{0\}$, and let $T \in \mathcal{B}(H)$ be self-adjoint. Then we have*

$$\|T\| = N_T.$$

Proof. It suffices to prove that $\|T\| \leq N_T$, hence that

$$\|T(x)\| \leq N_T \quad \text{for all } x \in H_1 = \{x \in H : \|x\| \leq 1\}. \quad (2.5.1)$$

We first note that if $v \in H$, then $|\langle T(v), v \rangle| \leq N_T \|v\|^2$.

Indeed, if $v = 0$, the claim is trivial. Otherwise, if $v \neq 0$ and $u := \frac{1}{\|v\|} v$, so $v = \|v\| u$, then

$$|\langle T(v), v \rangle| = \|v\|^2 |\langle T(u), u \rangle| \leq N_T \|v\|^2.$$

2. On Hilbert spaces and bounded linear operators

Let now $x \in H_1$. If $T(x) = 0$, then the inequality in (2.5.1) is trivially satisfied, so we can assume that $T(x) \neq 0$ and set $y := \frac{1}{\|T(x)\|} T(x) \in H_1$. Then we have

$$\|T(x)\| = \frac{1}{\|T(x)\|} \langle T(x), T(x) \rangle = \langle T(x), y \rangle. \quad (2.5.2)$$

Similarly, $\|T(x)\| = \langle y, T(x) \rangle$. As T is self-adjoint, we get

$$\|T(x)\| = \langle T(y), x \rangle. \quad (2.5.3)$$

Combining (2.5.2) and (2.5.3), and using our previous observations, as well as the parallelogram law and the fact that $\|x\| \leq 1$, $\|y\| = 1$, we get

$$\begin{aligned} \|T(x)\| &= \frac{1}{2} (\langle T(x), y \rangle + \langle T(y), x \rangle) \\ &= \frac{1}{4} (\langle T(x+y), x+y \rangle - \langle T(x-y), x-y \rangle) \\ &\leq \frac{1}{4} N_T (\|x+y\|^2 + \|x-y\|^2) \\ &= \frac{1}{2} N_T (\|x\|^2 + \|y\|^2) \\ &\leq N_T. \end{aligned}$$

This shows that (2.5.1) is satisfied, as desired. ■

Having in mind the spectral theorem for symmetric real matrices, it is legitimate to wonder whether it could be true that every self-adjoint operator $T \in \mathcal{B}(H)$ is *diagonalizable* in the sense that there always exists an orthonormal basis for H consisting of eigenvectors for T . However, as the next example illustrates, a self-adjoint operator may not have any eigenvalue at all, so this can not be true in general.

Example 2.5.10. Let $H = L^2([0, 1])$ (with usual Lebesgue measure) and let $T = M_f$ be the self-adjoint operator in $\mathcal{B}(H)$ given by multiplication with the bounded continuous function $f(t) = t$ on $[0, 1]$, cf. Example 2.5.4. Then we leave it as an exercise (cf. Exercise 2.30) to verify that T has no (complex) eigenvalues. □

We will see in Chapter 4 that every *compact* self-adjoint operator can be diagonalized in the sense mentioned above. Theorem 2.5.9 will help us to make the first step in this direction, by showing that a compact self-adjoint operator T has at least one an eigenvalue, namely $\|T\|$ or $-\|T\|$.

2.6 Unitary operators

In this section, we look at another important class of operators on Hilbert spaces. As a warm-up, we first characterize the linear operators which are isometric. If H is a Hilbert space, a map $T : H \rightarrow H$ is said to *preserve the inner product* if it satisfies that

$$\langle T(x), T(y) \rangle = \langle x, y \rangle \quad \text{for all } x, y \in H.$$

Proposition 2.6.1. *Let $H \neq \{0\}$ be a Hilbert space (over $\mathbb{K}$) and let $S : H \rightarrow H$. Then the following conditions are equivalent:*

- (i) $S \in \mathcal{B}(H)$ and $S^*S = I_H$;
- (ii) S is linear and preserves the inner product;
- (iii) S is a linear isometry.

Proof. (i) $\Rightarrow$ (ii): Assume $S \in \mathcal{B}(H)$ satisfies $S^*S = I_H$. Then S is linear and for all $x, y \in H$, we have

$$\langle S(x), S(y) \rangle = \langle x, (S^*S)(y) \rangle = \langle x, y \rangle,$$

so (ii) holds.

(ii) $\Rightarrow$ (iii): Any map preserving the inner product is isometric, so this is evident.

(iii) $\Rightarrow$ (i): Assume S is a linear isometry. Then $S \in \mathcal{B}(H)$ and $T := S^*S - I_H \in \mathcal{B}(H)$ is self-adjoint. Then for any $x \in H$, we have

$$\langle T(x), x \rangle = \langle (S^*S - I)(x), x \rangle = \langle S(x), S(x) \rangle - \langle x, x \rangle = \|S(x)\|^2 - \|x\|^2 = 0$$

Thus, $W_T = \{0\}$, so, using Theorem 2.5.9, we get that $\|T\| = N_T = 0$. Hence, $T = 0$, i.e., $S^*S = I_H$, so (i) holds. $\blacksquare$

Example 2.6.2. Assume H is finite-dimensional and $S : H \rightarrow H$ is a linear isometry, so $S^*S = I_H$, cf. Proposition 2.6.1. As S is injective, it is also surjective (since $\dim(S(H)) = \dim(H) - \dim(\ker(S)) = \dim(H)$, so $S(H) = H$). Thus, S is bijective, so it has an inverse S^{-1} (which is also a linear isometry). Since $S^*S = I_H$, we get that $S^{-1} = S^*$. In particular, we also have $SS^* = I_H$. $\square$

Remark 2.6.3. When H is infinite-dimensional, then a linear isometry S is not necessarily surjective. A typical example is the right shift operator S considered in Example 2.4.6, whose range does not contain the first basis vector; in this case, we have $S^*S = I_H$, while $SS^* \neq I_H$ (cf. Exercise 2.23). $\square$

2. On Hilbert spaces and bounded linear operators

Definition 2.6.4. Let H be a Hilbert space (over $\mathbb{K}$). An operator $U \in \mathcal{B}(H)$ is called *unitary* when it satisfies

$$U^*U = UU^* = I_H.$$

Thus, $U \in \mathcal{B}(H)$ is unitary if and only if U is bijective and $U^{-1} = U^*$.

When $\mathbb{K} = \mathbb{R}$, it is customary to say *orthogonal* instead of unitary.

Proposition 2.6.5. Let H be a Hilbert space (over $\mathbb{K}$) and let $U : H \rightarrow H$. Then the following conditions are equivalent:

- (i) $U \in \mathcal{B}(H)$ and U is unitary;
- (ii) U is bijective, linear and preserves the inner product;
- (iii) U is a surjective linear isometry.

Proof. (i) $\Rightarrow$ (ii): If $U \in \mathcal{B}(H)$ is unitary, then U is bijective and linear, and Proposition 2.6.1 gives that it preserves the inner product. Hence, (ii) holds.

(ii) $\Rightarrow$ (iii): This implication is evident.

(iii) $\Rightarrow$ (i): Suppose U is a surjective linear isometry. As a linear isometry is injective, U is bijective. Moreover, Proposition 2.6.1 gives that $U^*U = I_H$. So we get that $U^{-1} = U^*$, i.e., U is unitary, and (i) holds. $\blacksquare$

Example 2.6.6. Assume H has a countably infinite orthonormal basis $\mathcal{B}$ and D is the diagonal operator associated to a bounded sequence $\{\lambda_j\}_{j \in \mathbb{N}}$ in $\mathbb{K}$ (w.r.t. $\mathcal{B}$).

Then it is straightforward to check that D is unitary if and only if $\overline{\lambda_j} \lambda_j = 1$, i.e., $|\lambda_j| = 1$, for all $j \in \mathbb{N}$. $\square$

Example 2.6.7. Let $(X, \mathcal{A}, \mu)$ be a measure space and set $H := L^2(X, \mathcal{A}, \mu)$. For $f \in L^\infty(X, \mathcal{A}, \mu)$, consider the multiplication operator $M_f \in \mathcal{B}(H)$. Then we clearly have

$$(M_f)^* M_f = M_{|f|^2} = M_f (M_f)^*,$$

so we see that M_f is unitary whenever $|f| = 1$ μ -a.e. The converse holds if μ is semifinite, cf. Exercise 2.31. $\square$

Example 2.6.8. Let $H = \ell^2(\mathbb{Z})$. We may then define the *bilateral forward shift operator* $U : H \rightarrow H$ by

$$[U(\xi)](j) = \xi(j-1) \quad \text{for all } \xi \in H \text{ and all } j \in \mathbb{Z}.$$

2.6. Unitary operators

Indeed, since $\sum_{j \in \mathbb{Z}} |\xi(j-1)|^2 = \sum_{j \in \mathbb{Z}} |\xi(j)|^2 < \infty$, we see that $U(\xi) \in H$ and $\|U(\xi)\|_2 = \|\xi\|_2$ for every $\xi \in H$. Thus U is isometric.

Clearly, U is also linear. Moreover, it is surjective: if $\eta \in H$, then we have $U(\xi) = \eta$, where ξ is defined by $\xi(j) := \eta(j+1)$ for every $j \in \mathbb{Z}$ (since we may here argue as above to see that $\xi \in H$).

We may now conclude from Proposition 2.6.5 that U is unitary. Its adjoint $U^* = U^{-1}$ is called *the bilateral backward shift operator* (on $H = \ell^2(\mathbb{Z})$). We note that if $\mathcal{B} = \{e_n\}_{n \in \mathbb{Z}}$ denotes the canonical basis of $H = \ell^2(\mathbb{Z})$ as in Example 2.3.4, then we have

$$U(e_n) = e_{n+1} \quad \text{and} \quad U^*(e_n) = e_{n-1} \quad \text{for all } n \in \mathbb{Z}. \quad \square$$

Example 2.6.9. Let $H = L^2(\mathbb{R}, \mathcal{A}, \mu)$, where $\mathcal{A}$ denote the Lebesgue-measurable subsets of $\mathbb{R}$ and μ is the usual Lebesgue measure on $\mathcal{A}$. For each $x \in \mathbb{R}$ and $f \in L^2(\mathbb{R}, \mathcal{A}, \mu)$, define $f_x : \mathbb{R} \rightarrow \mathbb{C}$ by $f_x(y) = f(y-x)$ for every $y \in \mathbb{R}$. Then f_x is Lebesgue-measurable, and, using the translation invariance of μ , we get that

$$\|f_x\|_2^2 = \int_{\mathbb{R}} |f_x(y)|^2 d\mu(y) = \int_{\mathbb{R}} |f(y-x)|^2 d\mu(y) = \int_{\mathbb{R}} |f(y')|^2 d\mu(y') = \|f\|_2^2.$$

It follows that the map $U_x : H \rightarrow H$ given by $U_x(f) = f_x$ for each f in $L^2(\mathbb{R}, \mathcal{A}, \mu)$ is a well-defined unitary operator on H , satisfying that $(U_x)^* = U_{-x}$ (and $U_{x+x'} = U_x U_{x'}$ for all $x, x' \in \mathbb{R}$). You are asked to provide all the missing details in Exercise 2.33. $\square$

Let now H, K be Hilbert spaces (over $\mathbb{K}$). A bijective, linear map U from H onto K which preserves the inner product is often called an *isomorphism of Hilbert spaces*. As in Proposition 2.6.5, one shows that it is equivalent to require that U is a surjective linear isometry, or that $U \in \mathcal{B}(H, K)$ is unitary in the sense that we have $U^*U = I_H$ and $UU^* = I_K$. (Here, $U^* \in \mathcal{B}(K, H)$ denotes the adjoint of U , cf. Remark 2.4.3). We will therefore say that H and K are *isomorphic as Hilbert spaces* when such a map $U : H \rightarrow K$ exists.

Theorem 2.6.10. *Let $H \neq \{0\}$ be a Hilbert space over $\mathbb{C}$, and let $\mathcal{B}$ be an orthonormal basis of H . Then H and $\ell^2(\mathcal{B})$ are isomorphic as Hilbert spaces.*

Proof. Let $x \in H$ and define $\hat{x} : \mathcal{B} \rightarrow \mathbb{C}$ by

$$\hat{x}(u) := \langle x, u \rangle \quad \text{for all } u \in \mathcal{B}.$$

2. On Hilbert spaces and bounded linear operators

Note that Parseval's identity says that $\sum_{u \in \mathcal{B}} |\hat{x}(u)|^2 = \|x\|^2$. In particular, we have $\hat{x} \in \ell^2(\mathcal{B})$ and $\|\hat{x}\| = \|x\|$. Thus we can define an isometric map $U : H \rightarrow \ell^2(\mathcal{B})$ by

$$U(x) = \hat{x} \quad \text{for all } x \in H.$$

It is elementary to check that U is linear. Moreover, U is surjective. Indeed, let $\xi \in \ell^2(\mathcal{B})$. As $\sum_{u \in \mathcal{B}} |\xi(u)|^2 < \infty$, the set

$$\mathcal{B}_\xi := \{u \in \mathcal{B} : \xi(u) \neq 0\}$$

must be countable. Let $\{u_j\}_{j \in N}$ be an enumeration of $\mathcal{B}_\xi$, where $N = \{1, \dots, n\}$ for some $n \in \mathbb{N}$ or $N = \mathbb{N}$. Note that if $N = \mathbb{N}$, we have $\sum_{j=1}^\infty |\xi(u_j)|^2 < \infty$, so Lemma 2.3.7 gives that $\sum_{j=1}^\infty \xi(u_j) u_j$ is convergent in H .

Thus we may define $x \in H$ by $x := \sum_{j \in N} \xi(u_j) u_j$, and we then have

$$\hat{x}(u) = \langle x, u \rangle = \sum_{j \in N} \xi(u_j) \langle u_j, u \rangle = \begin{cases} \xi(u_k) & \text{if } u = u_k \text{ for some } k \in N, \\ 0 & \text{if } u \in \mathcal{B} \setminus \mathcal{B}_\xi, \end{cases}$$

i.e., $\hat{x}(u) = \xi(u)$ for all $u \in \mathcal{B}$. Hence, $U(x) = \xi$, showing that U is surjective.

We can now conclude that U is an isomorphism of Hilbert spaces from H to $\ell^2(\mathcal{B})$, as we wanted to show. $\blacksquare$

Remark 2.6.11. Theorem 2.6.10 is also true when $H \neq \{0\}$ is a Hilbert space over $\mathbb{R}$, but one has then to replace $\ell^2(\mathcal{B})$ with the *real* ℓ^2 -space

$$\ell_{\mathbb{R}}^2(\mathcal{B}) := \left\{ \xi : \mathcal{B} \rightarrow \mathbb{R} : \sum_{u \in \mathcal{B}} |\xi(u)|^2 < \infty, \right\}$$

considered as a Hilbert space over $\mathbb{R}$.

Remark 2.6.12. If $H \neq \{0\}$ is a Hilbert space over $\mathbb{C}$, and $\mathcal{B}, \mathcal{B}'$ are both orthonormal bases of H , then we get from Theorem 2.6.10 that $\ell^2(\mathcal{B})$ and $\ell^2(\mathcal{B}')$ are isomorphic as Hilbert spaces. It can be shown that this implies that (and in fact is equivalent to) $\mathcal{B}$ and $\mathcal{B}'$ having the same cardinality, meaning that there exists a bijection between $\mathcal{B}$ and $\mathcal{B}'$. (A similar statement holds if $H \neq \{0\}$ is a Hilbert space over $\mathbb{R}$). $\square$

2.7 Exercises

In the exercises of this chapter, H always denotes a Hilbert space over $\mathbb{K}$, unless otherwise stated.

Exercise 2.1. Let X be an inner product space. Check that the parallelogram law and the polarization identities hold.

Exercise 2.2. Let H be a Hilbert space which is infinite-dimensional, i.e., is not finite-dimensional (as a vector space). Argue first that there exists an orthonormal sequence $\{x_n\}_{n \in \mathbb{N}}$ in H . Then use this sequence to show that the closed unit ball H_1 is not compact.

Exercise 2.3. Consider $X := \ell^\infty(\mathbb{N})$ as a metric space w.r.t. $d(f, g) = \|f - g\|_u$. Let A be the subset of X given by

$$A := \{a^{(N)} : N \in \mathbb{N}\},$$

where $a^{(N)}(n) = 1$ if $1 \leq n \leq N$ and $a^{(N)}(n) = 0$ if $n > N$.

a) Show that A is closed in X .

b) Let $x \in X$ be given by $x(n) = 1 + 1/n$ for all $n \in \mathbb{N}$.

Show that $d(x, A) = 1$ and that $1 < d(x, a^{(N)})$ for all $N \in \mathbb{N}$.

Exercise 2.4. Let $c \in H$, $r > 0$ and set $B := B_r(c) = \{y \in H : \|y - c\| \leq r\}$. Check that B is closed and convex, and give a formula for x_B when $x \in H \setminus B$.

Exercise 2.5. Let S denote a nonempty subset of H .

a) Show that $S^\perp$ is a closed subspace of H .

b) Set $M := \overline{\text{Span}(S)}$. Verify that $S^\perp = M^\perp$. Then deduce that $M = (S^\perp)^\perp$. Deduce also that if N is a subspace of H , then $\overline{N} = (N^\perp)^\perp$.

Exercise 2.6. Let M be a closed subspace of H and $x \in H$.

a) Check that the associated map $P_M : H \rightarrow H$ is linear.

b) Show that $P_M(x) = y$ for some $y \in M$ if and only if $x - y \in M^\perp$.

Then show that $P_M(x)$ is the unique vector y in M such that $x - y \in M^\perp$.

Exercise 2.7. Assume $P \in \mathcal{B}(H)$ satisfies that $P^2 = P$ and $\|P\| = 1$.

Show that $P(H)$ is closed and $H = P(H) \oplus \ker P$. Then show that P is the orthogonal projection of H on $M := P(H)$.

Exercise 2.8. Let $(X, \mathcal{A}, \mu)$ be a measure space and $E \in \mathcal{A}$.

Set $\mathcal{A}_E = \{A \cap E : A \in \mathcal{A}\}$ and let $\mu_E = \mu|_{\mathcal{A}_E}$ denote the restriction of μ to $\mathcal{A}_E$. As should be well-known, $(E, \mathcal{A}_E, \mu_E)$ is a measure space.

Show that there exists an isometric isomorphism from $L^2(E, \mathcal{A}_E, \mu_E)$ onto the space M_E defined in Example 2.2.9, i.e.,

$$M_E = \{g \in L^2 : g \text{ lives essentially on } E\}.$$

2. On Hilbert spaces and bounded linear operators

Exercise 2.9. Consider $H = L^2([a, b], \mathcal{A}, \mu)$, where $\mathcal{A}$ denotes the σ -algebra of all Lebesgue measurable subsets of $[a, b]$, and μ is the usual Lebesgue measure on $\mathcal{A}$. Set

$$M := \left\{ g \in H : \int_{[a,b]} g \, d\mu = 0 \right\}.$$

Check that M is a closed subspace of H . Then, given $f \in H$, find an expression for the best approximation of f in M .

Exercise 2.10. Let H_1, H_2 be Hilbert spaces over $\mathbb{K}$ and consider

$$H := H_1 \times H_2 = \{(x_1, x_2) : x_1 \in H_1, x_2 \in H_2\}$$

as a vector space over $\mathbb{K}$ when equipped with its natural pointwise operations. For $(x_1, x_2), (y_1, y_2) \in H$, set

$$\langle (x_1, x_2), (y_1, y_2) \rangle := \langle x_1, y_1 \rangle + \langle x_2, y_2 \rangle.$$

a) Check that this gives an inner product on H such that H is a Hilbert space. Check also that the associated norm on H is given by

$$\|(x_1, x_2)\| = (\|x_1\|^2 + \|x_2\|^2)^{1/2}.$$

b) Set $\widetilde{H}_1 := \{(x_1, 0) : x_1 \in H_1\}$ and $\widetilde{H}_2 := \{(0, x_2) : x_2 \in H_2\}$,

Check that $\widetilde{H}_1$ and $\widetilde{H}_2$ are closed subspaces of H and that $\widetilde{H}_j$ is isomorphic to H_j ($j = 1, 2$).

Check also that $H = \widetilde{H}_1 \oplus \widetilde{H}_2$ and describe the orthogonal projection from H on $\widetilde{H}_1$ (resp. $\widetilde{H}_2$).

Note: The Hilbert space H is called the *external direct sum of H_1 and H_2* .

Exercise 2.11. Let $H \neq \{0\}$. Show that the following conditions are equivalent:

- (a) H is separable (i.e., H contains a countable dense subset);
- (b) H contains a countable subset S such that $\text{span}(S)$ is dense in H ;
- (c) H has a countable orthonormal basis.

Exercise 2.12. In the context of Fourier analysis described in Example 2.3.2) and Example 2.3.12, the formula

$$\|f\|_2 = \left(\sum_{n \in \mathbb{Z}} |\widehat{f}(n)|^2 \right)^{1/2},$$

is a special case of Parseval's identity stated in Theorem 2.3.8 c).

a) Set $f(t) = t$ for all $t \in [-\pi, \pi]$. Compute the Fourier coefficients of f . Then use Parseval's identity to show that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.

b) Set $g(t) = e^t$ for all $t \in [-\pi, \pi]$. Use Parseval's identity to obtain a formula for the sum of the series $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$.

Exercise 2.13. Let $H = L^2([-1, 1], \mathcal{A}, \mu)$, where $\mathcal{A}$ denote the Lebesgue-measurable subsets of $[-1, 1]$ and μ is the restriction of the usual Lebesgue measure to $\mathcal{A}$.

For each $n \in \{0\} \cup \mathbb{N}$, let $p_{n+1} : [-1, 1] \rightarrow \mathbb{C}$ be defined by $p_{n+1}(t) = t^n$, and set $S := \{[p_{n+1}] : n \in \{0\} \cup \mathbb{N}\} \subseteq H$.

a) Show that $\text{Span}(S)$ is dense in H .

b) Apply the Gram-Schmidt orthonormalization process to S to obtain an orthonormal basis $\mathcal{B} = \{[q_{n+1}] : n \in \{0\} \cup \mathbb{N}\}$ for H , where each q_{n+1} is the polynomial on $[-1, 1]$ given by

$$q_{n+1}(t) = \frac{\sqrt{n + \frac{1}{2}}}{2^n n!} \frac{d^n}{dt^n} ((t^2 - 1)^n).$$

(These polynomials are called *the normalized Legendre polynomials*.)

Exercise 2.14. Let H_1, H_2 be Hilbert spaces over $\mathbb{K}$ and consider

$$H := H_1 \times H_2 = \{(x_1, x_2) : x_1 \in H_1, x_2 \in H_2\}$$

as a vector space over $\mathbb{K}$ when equipped with its natural pointwise operations. For $(x_1, x_2), (y_1, y_2) \in H$, set

$$\langle (x_1, x_2), (y_1, y_2) \rangle := \langle x_1, y_1 \rangle + \langle x_2, y_2 \rangle.$$

a) Check that this gives an inner product on H such that H is a Hilbert space. Check also that the associated norm on H is given by

$$\|(x_1, x_2)\| = (\|x_1\|^2 + \|x_2\|^2)^{1/2}.$$

b) Set $\widetilde{H}_1 := \{(x_1, 0) : x_1 \in H_1\}$ and $\widetilde{H}_2 := \{(0, x_2) : x_2 \in H_2\}$,

Check that $\widetilde{H}_1$ and $\widetilde{H}_2$ are closed subspaces of H and that $\widetilde{H}_j$ is isomorphic to H_j ($j = 1, 2$).

Check also that $H = \widetilde{H}_1 \oplus \widetilde{H}_2$ and describe the orthogonal projection from H on $\widetilde{H}_1$ (resp. $\widetilde{H}_2$).

Note: The Hilbert space H is called the *external direct sum* of H_1 and H_2 .

c) Assume $\mathcal{B}_1$ and $\mathcal{B}_2$ are orthonormal bases for H_1 and H_2 , respectively.

Find an orthonormal basis $\mathcal{B}$ for H .

2. On Hilbert spaces and bounded linear operators

Exercise 2.15. The concept of generalized sums can be used to provide an alternative way of describing Fourier expansions in Hilbert spaces.

Let X be a normed space, J be a nonempty set, $\{x_j\}_{j \in J}$ be a family of elements of X , and $x \in X$. Then one says that the generalized sum $\sum_{j \in J} x_j$ converges to x when the following holds: given $\varepsilon > 0$, there exists a finite subset $F_0 \subseteq J$ such that for all finite subsets F of J containing F_0 , we have

$$\left\| x - \sum_{j \in F} x_j \right\| < \varepsilon,$$

in which case we write

$$x = \sum_{j \in J} x_j.$$

Consider a Hilbert space $H \neq \{0\}$, $\mathcal{B}$ an o.n.b. for H , and $x \in H$.

a) Show that we have

$$x = \sum_{u \in \mathcal{B}} \langle x, u \rangle u.$$

b) Show also that if $M \neq \{0\}$ is a closed subspace of H and $\mathcal{C}$ is an orthonormal basis for M , then we have

$$P_M(x) = \sum_{v \in \mathcal{C}} \langle x, v \rangle v.$$

Exercise 2.16. Assume that $H \neq \{0\}$ is finite-dimensional and $T : H \rightarrow H$ is linear. Let $\mathcal{B}$ be an orthonormal basis for H , and let $[T]_{\mathcal{B}}$ denote the matrix of T relative to $\mathcal{B}$.

Show that $[T^*]_{\mathcal{B}} = ([T]_{\mathcal{B}})^*$.

Exercise 2.17. Let $T \in \mathcal{B}(H)$. Assume H_0 is a dense subspace of H which is invariant under T , that is, we have $T(H_0) \subseteq H_0$, and let $T_0 \in \mathcal{B}(H_0)$ denote the restriction of T to H_0 . Further, assume there exists some $S_0 \in \mathcal{B}(H_0)$ such that

$$\langle T_0(x), y \rangle = \langle x, S_0(y) \rangle \quad \text{for all } x, y \in H_0.$$

Show that $T^* = S$, where $S \in \mathcal{B}(H)$ is the unique extension of S_0 provided by Theorem 1.3.2.

Exercise 2.18. Show that the formula for $(T_K)^*$ in Example 2.4.8 is correct.

Hint : Consider $H_0 = C([a, b])$ and use Exercise 2.17.

Exercise 2.19. Prove Proposition 2.4.12.

Exercise 2.20. Let $v, w \in H$ and consider the linear operator $T_{v,w} : H \rightarrow H$ defined by

$$T_{v,w}(x) := \langle x, v \rangle w \quad \text{for all } x \in H.$$

Note that $T_{v,w}$ has rank one if $v, w \in H \setminus \{0\}$.

a) Show that $T_{v,w}$ is bounded with norm $\|T_{v,w}\| = \|v\| \|w\|$. Then show that $(T_{v,w})^* = T_{w,v}$.

b) Show that every $T \in \mathcal{B}(H)$ which has rank one is of the form $T = T_{v,w}$ for some $v, w \in H \setminus \{0\}$.

c) Assume $T \in \mathcal{B}(H)$ is a finite-rank operator, $T \neq 0$. Show that T may be written as a finite sum of rank one operators in $\mathcal{B}(H)$.

Hint : Start by picking an orthonormal basis for $T(H)$.

d) Show that if $T \in \mathcal{B}(H)$ is a finite-rank operator, then so is T^* .

Exercise 2.21. Let $T \in \mathcal{B}(H)$ and let M be a closed subspace of H . Recall that M is said to be invariant under T when $T(M) \subseteq M$. Show that

M is invariant under T if and only if $M^\perp$ is invariant under T^* .

Exercise 2.22. Let H and K be Hilbert spaces over $\mathbb{K}$, and let $T \in \mathcal{B}(H, K)$.

a) Show that there exists a unique operator $T^* \in \mathcal{B}(K, H)$ (called the adjoint of T) satisfying that

$$\langle T(x), y \rangle = \langle x, T^*(y) \rangle \quad \text{for all } x \in H \text{ and all } y \in K.$$

b) Let $T' \in \mathcal{B}(H, K)$ and $\alpha, \beta \in \mathbb{K}$. Let also L be a Hilbert space over $\mathbb{K}$, and let $S \in \mathcal{B}(K, L)$, so that $ST \in \mathcal{B}(H, L)$. Show that the following properties hold:

- *i)* $(\alpha T + \beta T')^* = \bar{\alpha} T^* + \bar{\beta} T'^*$; *ii)* $(ST)^* = T^* S^*$; *iii)* $(T^*)^* = T$;
- *iv)* $\|T^*\| = \|T\|$; *v)* $\|T^* T\| = \|T\|^2$.

Exercise 2.23. Let $T \in \mathcal{B}(H)$.

a) Show that $\ker(T) = \ker(T^* T)$ and $\overline{T^*(H)} = \overline{(T^* T)(H)}$.

b) Assume T is *normal*, i.e., satisfies $T^* T = T T^*$. Show that

$$\ker(T^*) = \ker(T) \quad \text{and} \quad \overline{T^*(H)} = \overline{T(H)}.$$

c) Assume T is normal and has an eigenvalue λ . Show that $\bar{\lambda}$ is an eigenvalue of T^* , and that $E_\lambda^{T^*} = E_\lambda^T$.

2. On Hilbert spaces and bounded linear operators

d) Assume H has a countably infinite orthonormal basis $\mathcal{B}$, enumerated as $\mathcal{B} = \{u_j\}_{j \in \mathbb{N}}$, and let $S \in \mathcal{B}(H)$ be the right shift operator (w.r.t. $\mathcal{B}$). Set $T = S^*$.

Check that $TT^* = S^*S = I_H$, while $T^*T = SS^* = P$, where P is the orthogonal projection of H on $\{u_1\}^\perp$. Deduce that S and T are not normal.

Further, check that 0 is an eigenvalue for T , while 0 is not an eigenvalue of $T^* = S$. (*Note:* This shows that the assertion in c) does not necessarily hold when T is not normal.)

Finally, show that S has no eigenvalues, while every λ satisfying $|\lambda| < 1$ is an eigenvalue of T .

Exercise 2.24. Let $(X, \mathcal{A}, \mu)$ be a measure space. One says that $(X, \mathcal{A}, \mu)$ is *semifinite* when the following condition holds: if $E \in \mathcal{A}$ and $\mu(E) = \infty$, then there exists $F \subseteq E$, $F \in \mathcal{A}$ such that $0 < \mu(F) < \infty$.

a) Show that $(X, \mathcal{A}, \mu)$ is semifinite whenever it is σ -finite.

Assume from now on that $(X, \mathcal{A}, \mu)$ is semifinite. Set $H := L^2(X, \mathcal{A}, \mu)$. Let $f \in L^\infty(X, \mathcal{A}, \mu)$ and consider the multiplication operator $M_f \in \mathcal{B}(H)$ defined in Example 2.5.4.

b) Show that $\|M_f\| = \|f\|_\infty$.

c) Show that if M_f is self-adjoint, then f is real-valued μ -a.e. (As observed in Example 2.4.7, the converse is true without any restriction on $(X, \mathcal{A}, \mu)$.)

Exercise 2.25. Assume $P \in \mathcal{B}(H)$ is a self-adjoint projection, i.e., it satisfies that $P^* = P = P^2$. Show that P is the orthogonal projection of H on $M := P(H)$ (which is a closed subspace of H).

Exercise 2.26. Let $H \neq \{0\}$.

a) Assume $T \in \mathcal{B}(H)$ is self-adjoint. Deduce from Theorem 2.5.9 that $T = 0$ if and only if $\langle T(x), x \rangle = 0$ for all $x \in H$.

b) Suppose $\mathbb{K} = \mathbb{R}$. Give an example with $H = \mathbb{R}^2$ showing that the equivalence in a) may fail when T is not self-adjoint.

c) Assume $\mathbb{K} = \mathbb{C}$ and let $T \in \mathcal{B}(H)$. Show that $T = 0$ if and only if $\langle T(x), x \rangle = 0$ for all $x \in H$.

Exercise 2.27. Show that the set $\mathcal{B}(H)_{sa} := \{T \in \mathcal{B}(H) : T^* = T\}$ is closed in $\mathcal{B}(H)$ (w.r.t. operator norm).

Exercise 2.28. Let $H \neq \{0\}$. If $T \in \mathcal{B}(H)$ is self-adjoint, we have seen that $W_T \subseteq \mathbb{R}$; of course, if $\mathbb{K} = \mathbb{R}$, this gives no information on T as this inclusion is then true for any T in $\mathcal{B}(H)$. We assume therefore in this exercise that $H \neq \{0\}$ is a Hilbert space over $\mathbb{C}$.

Let $T \in \mathcal{B}(H)$. Then show that the following assertions are equivalent:

- (i) T is self-adjoint;
- (ii) $W_T \subseteq \mathbb{R}$;
- (iii) $\langle T(x), x \rangle \in \mathbb{R}$ for all $x \in H$.

Exercise 2.29. A self-adjoint operator T in $\mathcal{B}(H)$ is called *positive* when

$$\langle T(x), x \rangle \geq 0 \quad \text{for all } x \in H, \quad (2.7.1)$$

in which case we write $T \geq 0$.

(We note that if $\mathbb{K} = \mathbb{C}$ and $T \in \mathcal{B}(H)$ satisfies (2.7.1), then T is automatically self-adjoint, as follows from the previous exercise.)

a) Let $S \in \mathcal{B}(H)$, and let $R \in \mathcal{B}(H)$ be self-adjoint.

Check that $S^*S \geq 0$ and $R^2 \geq 0$. Then show that

$$\|S\| \leq 1 \Leftrightarrow (I_H - S^*S) \geq 0.$$

b) Let M be a closed subspace of H . Check that $P_M \geq 0$.

c) Assume that $T, T' \in \mathcal{B}(H)$ are positive and $\lambda \in [0, \infty)$.

Check that $T + T'$ and λT are also positive.

d) Show that the set of positive operators in $\mathcal{B}(H)$ is closed in $\mathcal{B}(H)$.

Exercise 2.30. Let $H = L^2([0, 1])$ (with usual Lebesgue measure) and let $T = M_f$ be the self-adjoint operator in $\mathcal{B}(H)$ given by multiplication with the function $f(t) = t$ on $[0, 1]$, cf. Example 2.5.4. Show that T has no (complex) eigenvalues.

Exercise 2.31. Let $(X, \mathcal{A}, \mu)$ be a semifinite measure space (cf. Exercise 2.24), and let $f \in L^\infty(X, \mathcal{A}, \mu)$. Suppose that the multiplication operator M_f on $H = L^2(X, \mathcal{A}, \mu)$ is unitary. Then show that $|f| = 1$ μ -a.e. (As observed in Example 2.6.7, the converse statement is true without any restriction on $(X, \mathcal{A}, \mu)$.)

2. On Hilbert spaces and bounded linear operators

Exercise 2.32. Assume $H \neq \{0\}$ is separable (cf. Exercise 2.11) and infinite-dimensional. Let then $\mathcal{B}$ be an orthonormal basis for H indexed by $\mathbb{Z}$, say $\mathcal{B} = \{v_k\}_{k \in \mathbb{Z}}$. One may then define *the bilateral shift operator* $V : H \rightarrow H$ (w.r.t. $\mathcal{B}$) by

$$V(x) = \sum_{k \in \mathbb{Z}} \langle x, v_k \rangle v_{k+1} \quad \text{for all } x \in H, \text{ i.e., by}$$

$$V(x) = \lim_{n \rightarrow \infty} \sum_{k=-n}^n \langle x, v_k \rangle v_{k+1} \quad \text{for all } x \in H.$$

- a) Show that V is a unitary operator in $\mathcal{B}(H)$.
- b) Describe V as a multiplication operator when $H = L^2([-\pi, \pi])$ (with normalized Lebesgue measure μ) and $v_k(t) = e^{ikt}$ for every $k \in \mathbb{Z}$.
- c) Assume $\mathbb{K} = \mathbb{C}$. Let $U : H \rightarrow \ell^2(\mathbb{Z})$ denote the isomorphism of Hilbert spaces defined in the proof of Theorem 2.6.10. Show that UVU^* is the bilateral forward shift operator on $\ell^2(\mathbb{Z})$.

Exercise 2.33. Provide all the details missing in Example 2.6.9.

Exercise 2.34. Let T be a bounded operator on a Hilbert space $H \neq \{0\}$. Check that the following properties of W_T and N_T hold:

- (a) $W_{T^*} = \{\bar{\lambda} : \lambda \in W_T\}$; hence, $N_{T^*} = N_T$.
- (b) W_T contains all the possible eigenvalues of T .
- (c) $W_{\alpha T + \beta I_H} = \alpha W_T + \beta$ for all $\alpha, \beta \in \mathbb{K}$.
- (d) $W_{UTU^*} = W_T$, hence $N_{UTU^*} = N_T$, for every unitary $U \in \mathcal{B}(H)$.
- (e) W_T is a compact subset of $\mathbb{K}$ when H is finite-dimensional.

It can also be shown that W_T is a convex subset of $\mathbb{K}$. This result is called the Toeplitz-Hausdorff Theorem, but the proof is beyond the scope of these notes.

CHAPTER 3

On Banach spaces and bounded linear operators

3.1 Quotient spaces

When given a vector space (over $\mathbb{K}$) and a subspace M of X , a useful construction is to form the *quotient space* X/M . It is defined as follows. Recall that if $x \in X$, then

$$x + M := \{x + m : m \in M \subseteq X\}.$$

As a set, the quotient space X/M is given by

$$X/M := \{x + M : x \in X\}.$$

Note that we may define an equivalence relation $\sim_M$ on X by $x \sim_M x'$ if and only if $x' - x \in M$; the equivalence class of x is then precisely the set $x + M$, and we have that

$$x + M = x' + M \Leftrightarrow x' - x \in M.$$

We can organize X/M as a vector space (over $\mathbb{K}$) by setting

$$(x + M) + (y + M) := (x + y) + M, \quad \alpha(x + M) := (\alpha x) + M$$

for all $x, y \in X$, $\alpha \in \mathbb{K}$. We leave it as an easy, but tedious exercise to check that these operations are well-defined and do turn X/M into a vector space. The zero element in X/M is of course $0 + M = M$.

The canonical surjective map $Q : X \rightarrow X/M$ defined by

$$Q(x) := x + M \quad \text{for all } x \in X$$

is called the *quotient map*. It is obviously linear.

3. On Banach spaces and bounded linear operators

Example 3.1.1. Assume H is a Hilbert space (over $\mathbb{K}$) and M is *closed* subspace of H . Then it is not difficult to identify H/M .

Indeed, thinking of $x + M$ as a translate of M , and setting $x_M := P_M(x) \in M$, we see (make a drawing!) that we can pick $x - x_M \in M^\perp$ as a representative of $x + M$. Now, defining $T : H/M \rightarrow M^\perp$ by

$$T(x + M) := x - x_M \quad \text{for all } x \in H,$$

it is not difficult to verify that T is well-defined, linear and bijective. Hence H/M is isomorphic to $M^\perp$ as a vector space.

Note that, as $M^\perp$ is a normed space, we may equip H/M with a norm by setting

$$\|x + M\| := \|T(x + M)\| = \|x - x_M\| = \inf\{\|x - m\| : m \in M\}$$

for every $x \in H$. In other words, $\|x + M\|$ is the distance from x to M . Inspired by this formula, we will also get a norm on X/M whenever X is a normed space and M is a closed subspace of X .

Definition 3.1.2. Assume X is a normed space (over $\mathbb{K}$) and M is a subspace of X . Then, for all $x \in X$, we set

$$\|x + M\| := \inf\{\|x - m\| : m \in M\}.$$

Proposition 3.1.3. Assume X is a normed space (over $\mathbb{K}$) and M is a subspace of X . Then we have:

- a) The map $x + M \mapsto \|x + M\|$ from X/M to $[0, \infty)$ is a seminorm on X/M ; it gives a norm on X/M (called the quotient norm) if and only if M is closed.
- b) If M is closed, then the quotient map $Q : X \rightarrow X/M$ is bounded, with $\|Q\| \leq 1$.

Proof. a) We first check that the map $x + M \mapsto \|x + M\|$ is well-defined. Assume $x + M = x' + M$. Then $x' = x + m'$ for some $m' \in M$, so

$$\begin{aligned} \|x' + M\| &= \inf\{\|x' - m\| : m \in M\} = \inf\{\|x - (m - m')\| : m \in M\} \\ &= \inf\{\|x - m''\| : m'' \in M\} = \|x + M\|. \end{aligned}$$

Next we show that the triangle inequality holds. Consider $x + M, y + M \in X/M$. Let $\varepsilon > 0$. We may then find $m_1, m_2 \in M$ such that

$$\|x - m_1\| < \|x + M\| + \varepsilon/2, \quad \|y - m_2\| < \|y + M\| + \varepsilon/2.$$

As $m_1 + m_2 \in M$, we get (using the triangle inequality in X) that

$$\begin{aligned} \|(x + M) + (y + M)\| &= \|(x + y) + M\| \leq \|x + y - (m_1 + m_2)\| \\ &\leq \|x - m_1\| + \|y - m_2\| < \|x + M\| + \|y + M\| + \varepsilon \end{aligned}$$

Since this holds for all $\varepsilon > 0$, we get

$$\|(x + M) + (y + M)\| \leq \|x + M\| + \|y + M\|.$$

The homogeneity property is easily verified, so we skip the proof. Since the map $x + M \mapsto \|x + M\|$ takes non-negative values for each $x \in X$, we get that it is a seminorm on X/M .

Moreover, for $x \in X$, we have

$$x \in \overline{M} \Leftrightarrow \inf\{\|x - m\| : m \in M\} = 0 \Leftrightarrow \|x + M\| = 0,$$

and the last assertion in a) readily follows.

b) Assume M is closed. Then for all $x \in X$, we have

$$\|Q(x)\| = \|x + M\| = \inf\{\|x - m\| : m \in M\} \leq \|x\| \quad (\text{since } 0 \in M),$$

so Q is bounded with $\|Q\| \leq 1$. ■

Theorem 3.1.4. *Assume X is a Banach space (over $\mathbb{K}$) and M is a closed subspace of X . Then X/M is a Banach space (w.r.t. the quotient norm).*

Proof. Let $\{Z_n\}_{n \in \mathbb{N}}$ be a Cauchy sequence in X/M . To show that it is convergent (thereby showing that X/M is complete), it suffices to show that it has a convergent subsequence¹.

We first choose $n_1 \in \mathbb{N}$ such that

$$n, m \geq n_1 \Rightarrow \|Z_n - Z_m\| < \frac{1}{2}.$$

Then we choose inductively $n_1 < n_2 < \dots < n_i < \dots$ in $\mathbb{N}$ such that

$$i \in \mathbb{N} \text{ and } n, m \geq n_i \Rightarrow \|Z_n - Z_m\| < \frac{1}{2^i}.$$

For each $i \in \mathbb{N}$, set $Z'_i := Z_{n_i}$. By the above inequality we then have

$$\|Z'_{i+1} - Z'_i\| < \frac{1}{2^i} \quad \text{for all } i \in \mathbb{N}.$$

¹If you are not familiar with this fact from a previous course you should prove it.

3. On Banach spaces and bounded linear operators

Our goal is to show that $\{Z'_i\}_{i \in \mathbb{N}}$ is convergent. (Since $\{Z'_i\}_{i \in \mathbb{N}} = \{Z_{n_i}\}_{i \in \mathbb{N}}$ is a subsequence of $\{Z_n\}_{n \in \mathbb{N}}$, this will finish the proof.)

We first write $Z'_1 = x_1 + M$ for some $x_1 \in X$. Next, we write $Z'_2 = x'_2 + M$ for some $x'_2 \in X$. Since

$$\|(x'_2 - x_1) + M\| = \|(x'_2 + M) - (x_1 + M)\| = \|Z'_2 - Z'_1\| < \frac{1}{2},$$

we can find $m_2 \in M$ such that $\|(x'_2 - x_1) - m_2\| < \frac{1}{2}$.

We now set $x_2 := x'_2 - m_2 \in X$. Then we have

$$Z'_2 = x'_2 + M = x_2 + M \quad \text{and} \quad \|x_2 - x_1\| < \frac{1}{2}.$$

Proceeding inductively as above, we can find for each $i \geq 2$ some $x_{i+1} \in X$ such that

$$Z'_{i+1} = x_{i+1} + M \quad \text{and} \quad \|x_{i+1} - x_i\| < \frac{1}{2^i}.$$

Note that the sequence $\{x_i\}_{i \in \mathbb{N}}$ is then a Cauchy sequence in X . Indeed, consider $n > m \geq 1$, so $n = m + k$ for some $k \in \mathbb{N}$. Then

$$\begin{aligned} \|x_n - x_m\| &= \left\| \sum_{j=1}^k (x_{m+j} - x_{m+j-1}) \right\| \leq \sum_{j=1}^k \|x_{m+j} - x_{m+j-1}\| \\ &< \sum_{j=1}^k \frac{1}{2^{m+j-1}} = \frac{1}{2^m} \sum_{j=1}^k \frac{1}{2^{j-1}} < \frac{1}{2^{m-1}}. \end{aligned}$$

Since $\frac{1}{2^{m-1}} \rightarrow 0$ as $m \rightarrow \infty$, it is clear that we can make $\|x_n - x_m\|$ as small as we want by choosing m large enough.

Hence, the sequence $\{x_i\}_{i \in \mathbb{N}}$ is Cauchy. As X is complete, it converges to some $x \in X$. Setting $Z := x + M \in X/M$, we get that

$$\|Z'_i - Z\| = \|(x_i - x) + M\| \leq \|x_i - x\| \rightarrow 0 \text{ as } i \rightarrow \infty,$$

i.e., $\{Z'_i\}_{i \in \mathbb{N}}$ is convergent, as we wanted to show. ■

Quotient spaces are often useful in the study of linear maps.

Proposition 3.1.5. *Let $T : X \rightarrow Y$ be a linear map between vector spaces (over $\mathbb{K}$) and assume M is a subspace of X satisfying $M \subseteq \ker(T)$. Then the map $\tilde{T} : X/M \rightarrow Y$ given by*

$$\tilde{T}(x + M) := T(x) \quad \text{for all } x \in X$$

is well-defined and linear, and we have $T = \tilde{T} \circ Q$, where $Q : X \rightarrow X/M$ is the quotient map.

3.1. Quotient spaces

Proof. We first check that $\tilde{T}$ is well-defined. Assume $x + M = x' + M$ for $x, x' \in X$. Then $x' = x + m$ for some $m \in M$. Since $M \subseteq \ker(T)$, we have $T(m) = 0$. So

$$T(x') = T(x + m) = T(x) + T(m) = T(x).$$

Next, we check that $\tilde{T}$ is linear. Assume $x, x' \in X$ and $\alpha \in \mathbb{K}$. Then

$$\begin{aligned} \tilde{T}(\alpha(x + M) + (x' + M)) &= \tilde{T}((\alpha x + x') + M) = T(\alpha x + x') \\ &= \alpha T(x) + T(x') = \alpha \tilde{T}(x + M) + \tilde{T}(x' + M). \end{aligned}$$

The last assertion follows immediately from the definition of Q . ■

Corollary 3.1.6. *Let $T : X \rightarrow Y$ be a linear map between vector spaces (over $\mathbb{K}$). Then the vector spaces $X/\ker(T)$ and $T(X)$ are isomorphic.*

Proof. Applying Proposition 3.1.5 with $M = \ker(T)$, we get a linear map $\tilde{T} : X/\ker(T) \rightarrow Y$ whose range is clearly equal to $T(X)$.

Moreover, $\tilde{T}$ is injective. Indeed, assume $\tilde{T}(x + \ker(T)) = \tilde{T}(x' + \ker(T))$ for $x, x' \in X$. Then $T(x) = T(x')$, so $T(x' - x) = 0$. Hence, $x' - x \in \ker(T)$, i.e., $x + \ker(T) = x' + \ker(T)$.

Thus, $\tilde{T}$, considered as a map from $X/\ker(T)$ to $T(X)$, is an isomorphism of vector spaces. ■

We now look at the case where T is a bounded linear map.

Proposition 3.1.7. *Let X, Y be normed spaces (over $\mathbb{K}$) and M be a closed subspace of X , so X/M is a normed space w.r.t. the quotient norm. Assume $T \in \mathcal{B}(X, Y)$ satisfies $M \subseteq \ker(T)$, and consider the linear map $\tilde{T} : X/M \rightarrow Y$ given by*

$$\tilde{T}(x + M) = T(x) \quad \text{for all } x \in X.$$

Then $\tilde{T}$ is bounded, with $\|\tilde{T}\| = \|T\|$.

Proof. Let $x \in X$. For every $m \in M$, we have $T(m) = 0$, so

$$\|\tilde{T}(x + M)\| = \|T(x)\| = \|T(x - m)\| \leq \|T\| \|x - m\|.$$

Hence,

$$\|\tilde{T}(x + M)\| \leq \|T\| \inf\{\|x - m\| : m \in M\} = \|T\| \|x + M\|.$$

This shows that $\tilde{T}$ is bounded, with $\|\tilde{T}\| \leq \|T\|$. Further, for all $x \in X$, we have

$$\|T(x)\| = \|\tilde{T}(x + M)\| \leq \|\tilde{T}\| \|x + M\| \leq \|\tilde{T}\| \|x\|.$$

Thus, $\|T\| \leq \|\tilde{T}\|$, so $\|T\| = \|\tilde{T}\|$. ■

3. On Banach spaces and bounded linear operators

Remark 3.1.8. Let M be a closed subspace of a normed space X . Set

$$M^\perp := \{\varphi \in X^* : \varphi = 0 \text{ on } M\}.$$

It is easy to check that $M^\perp$ is a closed subspace of X^* , which is called the *annihilator* of M . Moreover, it can be shown that

- $(X/M)^*$ is isometrically isomorphic to $M^\perp$.
- M^* is isometrically isomorphic to $X^*/M^\perp$.

We will give a proof of these statements in Theorem 3.6.14.

3.2 The Baire category theorem

As Baire's so-called category theorem is not always covered in courses in real analysis, we include a proof. Let (X, d) denote a metric space. For $x \in X$ and $\varepsilon > 0$, we set $B(x, \varepsilon) := \{y \in X : d(x, y) < \varepsilon\}$. We recall some relevant definitions.

Let A be a nonempty subset of X . The *diameter* of A is given by

$$\text{diam}(A) := \sup\{d(x, y) : x, y \in A\} \in [0, \infty].$$

One easily checks that $\text{diam}(A)$ is finite if and only if A is bounded. Moreover, A is said to be *dense* in X when $\overline{A} = X$, i.e., for every $x \in X$ and $\varepsilon > 0$, we have $A \cap B(x, \varepsilon) \neq \emptyset$. Equivalently, $A \cap U \neq \emptyset$ for every nonempty open subset U of X . The following result, which is sometimes called *Cantor's theorem*, will be helpful to prove Baire's theorem.

Theorem 3.2.1. *Assume (X, d) is complete and $\{A_n\}_{n \in \mathbb{N}}$ is sequence of nonempty closed subsets of X satisfying $A_{n+1} \subseteq A_n$ for every $n \in \mathbb{N}$ and $\lim_{n \rightarrow \infty} \text{diam}(A_n) = 0$. Then $\bigcap_{n=1}^{\infty} A_n$ is nonempty and consists of a single point.*

Proof. For each $n \in \mathbb{N}$, pick $a_n \in A_n$. Then the sequence $\{a_n\}_{n \in \mathbb{N}}$ is Cauchy. Indeed, let $\varepsilon > 0$. Choose $N \in \mathbb{N}$ such that $\text{diam}(A_n) < \varepsilon$ for all $n \geq N$. Then for all $m, n \geq N$ we have $a_m, a_n \in A_N$, so $d(a_m, a_n) < \varepsilon$. Since (X, d) is complete, $\{a_n\}_{n \in \mathbb{N}}$ converges to some $a \in X$. Then for each $n \in \mathbb{N}$, a is the limit of the subsequence $\{a_k : k \geq n\}$ which lies in A_n , so $a \in \overline{A_n} = A_n$. Hence, $a \in \bigcap_{n=1}^{\infty} A_n$. Now assume that there exists some $b \in \bigcap_{n=1}^{\infty} A_n$ such that $b \neq a$. Then for each $n \in \mathbb{N}$, we have $\text{diam}(A_n) \geq d(a, b)$, so we get

$$\lim_{n \rightarrow \infty} \text{diam}(A_n) \geq d(a, b) > 0$$

contradicting that $\lim_{n \rightarrow \infty} \text{diam}(A_n) = 0$. Thus $\bigcap_{n=1}^{\infty} A_n = \{a\}$. ■

3.2. The Baire category theorem

Theorem 3.2.2 (Baire). *Assume (X, d) is complete and $\{U_n\}_{n \in \mathbb{N}}$ is a sequence of open dense subsets of X . Then $\bigcap_{n=1}^{\infty} U_n$ is dense in X .*

Proof. Set $U := \bigcap_{n=1}^{\infty} U_n$, and let $x \in X$, $\varepsilon > 0$. We have to show that $U \cap B(x, \varepsilon) \neq \emptyset$.

Consider first $z \in X$, $r > 0$ and $n \in \mathbb{N}$. As U_n is dense in X , we can pick some $y \in U_n \cap B(z, r)$. Now, $U_n \cap B(z, r)$ is open, so we can find some $s > 0$ such that $B(y, s) \subseteq U_n \cap B(z, r)$. We then have

$$\overline{B(y, r')} \subseteq B(y, s) \subseteq U_n \cap B(z, r) \quad \text{for all } 0 < r' < s.$$

Using this observation (with x, ε and $n = 1$), we can find $x_1 \in U_1$ and $r_1 < 1/2$ such that

$$\overline{B(x_1, r_1)} \subseteq U_1 \cap B(x, \varepsilon).$$

Next, using it again (now with x_1, r_1 and $n = 2$), we can find $x_2 \in U_2$ and $r_2 < 1/4$ such that

$$\overline{B(x_2, r_2)} \subseteq U_2 \cap B(x_1, r_1)$$

. Proceeding inductively, we can find for each $n \geq 1$ some $x_{n+1} \in U_{n+1}$ and $r_{n+1} < 2^{-(n+1)}$ such that

$$\overline{B(x_{n+1}, r_{n+1})} \subseteq U_{n+1} \cap B(x_n, r_n).$$

We then have

$$\overline{B(x_{n+1}, r_{n+1})} \subseteq B(x_n, r_n) \subseteq \overline{B(x_n, r_n)}$$

for every $n \geq 1$, and $\text{diam}(\overline{B(x_n, r_n)}) = 2r_n < 2 \cdot 2^{-n} \rightarrow 0$ as $n \rightarrow \infty$.

Using Cauchy's theorem (with $A_n = \overline{B(x_n, r_n)}$), we can conclude that there exists (a unique) $a \in \bigcap_{n=1}^{\infty} \overline{B(x_n, r_n)}$. Then we have

$$a \in \bigcap_{n=1}^{\infty} \overline{B(x_n, r_n)} \subseteq \bigcap_{n=1}^{\infty} U_n = U \text{ and } a \in U_1 \cap B(x, \varepsilon) \subseteq B(x, \varepsilon).$$

Thus, $a \in U \cap B(x, \varepsilon)$, so $U \cap B(x, \varepsilon) \neq \emptyset$, as desired. ■

A subset N of X is called *nowhere dense* if $\overline{N}$ has empty interior, i.e., $\overline{N}^0 = \emptyset$. In particular, a closed subset of X is nowhere dense if it has empty interior. A subset M of X is called *meager* if it can be written as a countable union of nowhere dense subsets of X . (Baire himself called meager sets for sets of the first category, second category sets being those not of the first category).

3. On Banach spaces and bounded linear operators

Corollary 3.2.3. *A complete metric space (X, d) is not meager.*

Proof. Assume that X is meager, so we may write $X = \cup_{k \in K} N_k$, where K is countable and each N_k is nowhere dense. We may assume $K = \mathbb{N}$ (for if K is finite, say $K = \{k_1, \dots, k_n\}$, then $X = \cup_{j=1}^n N'_{k_j}$, where $N'_j = N_{k_j}$ if $1 \leq j \leq n$, and $N'_j = \emptyset$ if $j > n$).

For each $k \in \mathbb{N}$, set $U_k := X \setminus \overline{N_k}$. Then each U_k is open, and dense since $\overline{U_k} = X \setminus \overline{N_k}^0 = X \setminus \emptyset = X$. We also have

$$\bigcap_{k=1}^{\infty} U_k = \bigcap_{k=1}^{\infty} (X \setminus \overline{N_k}) = X \setminus \left(\bigcup_{k=1}^{\infty} \overline{N_k} \right) = X \setminus X = \emptyset,$$

which is obviously not dense in X . Thus, Baire's theorem implies that (X, d) is not complete. ■

It follows for example that $\mathbb{R}$ is uncountable. Indeed, assume (for contradiction) that $\mathbb{R}$ was countable, say $\mathbb{R} = \{x_k : k \in K\}$ with K countable. As each one-point set $\{x\}$ is nowhere dense in $\mathbb{R}$ (w.r.t. its standard metric), we could then conclude that $\mathbb{R} = \cup_{k \in K} \{x_k\}$ is meager, contradicting the completeness of $\mathbb{R}$.

3.3 The principle of uniform boundedness

Assume $\{T_n\}_{n \in \mathbb{N}}$ is a sequence of bounded linear maps between two normed spaces X and Y , satisfying that $\{T_n(x)\}_{n \in \mathbb{N}}$ converges in Y for each x in X . We can then define a linear map $T : X \rightarrow Y$ by $T(x) = \lim_{n \rightarrow \infty} T_n(x)$ for every $x \in X$. A natural question is then: is T necessarily bounded? Using the principle of uniform boundedness we will see that the answer is always positive when X is a Banach space.

The proof relies on *the principle of uniform boundedness*, stated below, which is sometimes called *the Banach-Steinhaus theorem*.

Theorem 3.3.1. *Let X be a Banach space and Y be a normed space (both over $\mathbb{K}$). Assume $\{T_i\}_{i \in I}$ is a family of operators in $\mathcal{B}(X, Y)$ such that*

$$\sup \left\{ \|T_i(x)\| : i \in I \right\} < \infty \text{ for all } x \in X.$$

Then $\{T_i\}_{i \in I}$ is uniformy bounded, i.e., $\sup \left\{ \|T_i\| : i \in I \right\} < \infty$.

Proof. Let $n \in \mathbb{N}$ and $i \in I$. By continuity of T_i and of the norm in Y , the function $f : X \rightarrow \mathbb{R}$ mapping each x to $\|T_i(x)\|$ is continuous. Hence the set

$$\left\{ x \in X : \|T_i(x)\| \leq n \right\} = f^{-1}([0, n])$$

3.3. The principle of uniform boundedness

is closed in X . Since an intersection of closed sets is closed, we get that each set

$$F_n := \left\{ x \in X : \sup\{\|T_i(x)\| : i \in I\} \leq n \right\} = \bigcap_{i \in I} \left\{ x \in X : \|T_i(x)\| \leq n \right\}$$

is closed in X . Now the assumption gives that $X = \bigcup_{n=1}^{\infty} F_n$. Hence at least one of the F_n 's, say F_N , must have non-empty interior: otherwise, X would be meager, contradicting that X is complete, cf. Corollary 3.2.3. Thus there exist $x_0 \in F_N$ and $r > 0$ such that $B(x_0, r) \subseteq F_N$.

Set $s := r/2$ and consider $x \in X$ such that $\|x\| \leq s$. Then

$$x_0 - x \in B(x_0, r) \subseteq F_N,$$

so we get

$$\|T_i(x)\| = \|T_i(x_0 - (x_0 - x))\| \leq \|T_i(x_0)\| + \|T_i(x_0 - x)\| \leq N + N = 2N$$

for each $i \in I$. Hence, for every $x \in X$ such that $\|x\| \leq 1$, we have

$$\|T_i(x)\| = \frac{1}{s} \|T_i(sx)\| \leq \frac{2N}{s},$$

showing that $\|T_i\| \leq 2N/s$ for each $i \in I$. It follows that

$$\sup \left\{ \|T_i\| : i \in I \right\} \leq \frac{2N}{s} < \infty.$$

■

Corollary 3.3.2. *Let X be a Banach space and Y be a normed space (both over $\mathbb{K}$). Assume $\{T_n\}_{n \in \mathbb{N}}$ is a sequence in $\mathcal{B}(X, Y)$ satisfying that $\{T_n(x)\}_{n \in \mathbb{N}}$ converges in Y for each x in X . Then the linear operator $T : X \rightarrow Y$ defined by*

$$T(x) = \lim_{n \rightarrow \infty} T_n(x) \text{ for every } x \in X$$

is bounded.

Proof. Let $x \in X$. Since the sequence $\{T_n(x)\}_{n \in \mathbb{N}}$ is convergent, it is bounded, so $\sup \{\|T_n(x)\| : n \in \mathbb{N}\} < \infty$. As this holds for every $x \in X$, the principle of uniform boundedness gives that $M := \sup \{\|T_n\| : n \in \mathbb{N}\} < \infty$. We then have

$$\|T(x)\| = \lim_{n \rightarrow \infty} \|T_n(x)\| \leq \sup \{\|T_n(x)\| : n \in \mathbb{N}\} \leq M \|x\|$$

for every $x \in X$, i.e., T is bounded. ■

3. On Banach spaces and bounded linear operators

Remark 3.3.3. Note that the sequence $\{T_n\}_{n \in \mathbb{N}}$ in Corollary 3.3.2 will not necessarily converge to T in operator norm. Indeed, consider for example the Banach space $(c_0(\mathbb{N}), \|\cdot\|_\infty)$ and $T_n \in \mathcal{B}(c_0(\mathbb{N}), \mathbb{C})$ given by $T_n(f) = f(n)$ for $n \in \mathbb{N}$ and $f \in c_0(\mathbb{N})$. Letting $T \in \mathcal{B}(c_0(\mathbb{N}), \mathbb{C})$ be the zero map, we have $\lim_{n \rightarrow \infty} T_n(f) = 0 = T(f)$ for every $f \in c_0(\mathbb{N})$. On the other hand, we have $\|T_n\| = 1$ for each n , so $\{T_n\}$ does not converge to T in operator norm. However, a weaker statement may be shown (cf. Exercise 3.10).

Example 3.3.4. Let $\mathcal{B} = \{e_k : k \in \mathbb{Z}\}$ be the orthonormal basis for $H = L^2([-\pi, \pi])$ described in Example 2.3.2. As we have seen in Example 2.3.12, the Fourier series $\sum_{k \in \mathbb{Z}} \hat{f}(k) e_k$ converges to f w.r.t. $\|\cdot\|_2$ for every $f \in H$. Set

$$X := \{f \in C([-\pi, \pi]) : f(-\pi) = f(\pi)\},$$

which is a closed subspace of $C([-\pi, \pi])$ w.r.t. $\|\cdot\|_\infty$, hence a Banach space, that may be identified with the space of continuous 2π -periodic complex functions on $\mathbb{R}$. It is natural to wonder whether the Fourier series of f converges to f w.r.t. $\|\cdot\|_\infty$ whenever $f \in X$. (It should be well-known from a previous course that this happens under some smoothness condition on f .) Now, one may deduce from the principle of uniform boundedness that there must exist some $f \in X$ for which the Fourier series fails to converge pointwise on $[-\pi, \pi]$. In particular, the Fourier series of such a function f can not converge to f uniformly on $[-\pi, \pi]$.

We sketch the argument (and leave to the reader to provide the missing details). For $f \in X$ and $n \in \mathbb{N}$, let $f_n \in X$ be given by $f_n = \sum_{k=-n}^n \hat{f}(k) e_k$. An elementary computation gives that

$$f_n(0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) D_n(t) dt, \quad \text{where } D_n(t) := \sum_{k=-n}^n e^{ikt}, \quad |t| \leq \pi.$$

Consider now the linear map $T_n : X \rightarrow \mathbb{C}$ defined by $T_n(f) = f_n(0)$ for all $f \in X$. It is not difficult to check that each T_n is bounded with $\|T_n\| = \frac{1}{2\pi} \int_{-\pi}^{\pi} |D_n(t)| dt$. Using that

$$D_n(t) = \begin{cases} \frac{\sin((n+1/2)t)}{\sin(t/2)} & \text{if } t \neq 0, \\ 2n+1 & \text{if } t = 0 \end{cases} \quad \text{for } |t| \leq \pi,$$

and $|\sin(t/2)| \leq |t|/2$, it can be shown that for some $M > 0$ we have

$$\|T_n\| \geq M \sum_{k=1}^n \frac{1}{k}$$

for every $n \in \mathbb{N}$. As the harmonic series diverges, this implies that $\|T_n\| \rightarrow \infty$ as $n \rightarrow \infty$. Thus, $\sup\{\|T_n\| : n \in \mathbb{N}\} = \infty$. The principle of uniform

boundedness gives that there must exist at least one $f \in X$ such that $\sup\{|T_n(f)| : n \in \mathbb{N}\} = \infty$, i.e., $\sup\{|f_n(0)| : n \in \mathbb{N}\} = \infty$. This implies that the sequence $\{f_n(0)\}_{n \in \mathbb{N}}$ is not convergent, hence that $\{f_n(0)\}_{n \in \mathbb{N}}$ does not converge to $f(0)$. So $\{f_n\}$ does not converge pointwise to f on $[-\pi, \pi]$.

3.4 The open mapping theorem

We recall that a map $f : X \rightarrow Y$ between two metric spaces X and Y is continuous if and only if $f^{-1}(V)$ is an open subset of X for every open subset V of Y . On the other hand, the map f is called *open* if $f(U)$ is an open subset of Y for every open subset U of X . Thus, if $g : X \rightarrow Y$ is bijective, then g is open if and only if $g^{-1} : Y \rightarrow X$ is continuous.

As another consequence of the Baire category theorem, we will prove the *open mapping theorem*, which says that any surjective bounded operator between Banach spaces is an open map.

Theorem 3.4.1. *Let X, Y be Banach spaces (over $\mathbb{K}$) and $T \in \mathcal{B}(X, Y)$. Assume T is surjective. Then T is an open map from X to Y .*

The proof is somewhat involved, so we will divide it several steps. First we will establish a criterion for a bounded operator to be open. We will use the following notation. If X is a vector space over $\mathbb{K}$, $x \in X$, $A \subseteq X$ and $\lambda \in \mathbb{K}$, then we set

$$x + A := \{x + a : a \in A\} \quad \text{and} \quad \lambda A := \{\lambda a : a \in A\}.$$

Moreover, if X is a normed space and $r > 0$, we will denote the closed ball with center at x and radius r by

$$B[x, r] := \{y \in X : \|y - x\| \leq r\}.$$

Lemma 3.4.2. *Let X, Y be normed spaces (over $\mathbb{K}$) and $T \in \mathcal{B}(X, Y)$. Assume there exists some $\delta > 0$ such that*

$$B[0, \delta] \subseteq T(B(0, 1)).$$

Then T is open.

Proof. Let $U \subseteq X$ be open. We have to show that $T(U)$ is open in Y , i.e., $T(x_0)$ is an interior point of $T(U)$ for every $x_0 \in U$.

3. On Banach spaces and bounded linear operators

So let $x_0 \in U$, and choose $s > 0$ such that $B(x_0, s) \subseteq U$. Then, using the assumption at the second step, we get

$$\begin{aligned} B[T(x_0), s\delta] &= T(x_0) + sB[0, \delta] \\ &\subseteq T(x_0) + sT(B(0, 1)) \\ &= T(x_0) + T(B(0, s)) \\ &= T(B(x_0, s)) \\ &\subseteq T(U) \end{aligned}$$

Thus, choosing $0 < r < s\delta$, we get that $B(T(x_0), r) \subseteq T(U)$. Hence, $T(x_0) \in T(U)^0$, as desired. $\blacksquare$

Lemma 3.4.3. *Let X, Y be normed spaces (over $\mathbb{K}$) and $T \in \mathcal{B}(X, Y)$. Assume there exist some $y \in Y$ and $r > 0$ such that*

$$B[y, r] \subseteq \overline{T(B(0, 1))}.$$

Then we have

$$B[0, r] \subseteq \overline{T(B(0, 1))}.$$

Proof. Set $Z := T(B(0, 1)) \subseteq Y$. We first make an observation. Assume $w, w' \in \overline{Z}$. Then we claim that

$$-w \in \overline{Z} \quad \text{and} \quad \frac{1}{2}(w + w') \in \overline{Z}.$$

Indeed, pick sequences $\{x_n\}$ and $\{x'_n\}$ in $B(0, 1)$ such that $\lim_{n \rightarrow \infty} T(x_n) = w$ and $\lim_{n \rightarrow \infty} T(x'_n) = w'$. Then $\{-x_n\}$ and $\{\frac{1}{2}(x_n + x'_n)\}$ are sequences in $B(0, 1)$ such that

$$\lim_{n \rightarrow \infty} T(-x_n) = -w \quad \text{and} \quad \lim_{n \rightarrow \infty} T\left(\frac{1}{2}(x_n + x'_n)\right) = \frac{1}{2}(w + w').$$

This proves the claim above.

Assume now that $v \in B[0, r]$. Note that $y \pm v \in B[y, r]$. The assumption gives that $y \pm v \in \overline{Z}$. Using our observation above, we get that

$$-y + v = -(y - v) \in \overline{Z}, \quad \text{and} \quad v = \frac{1}{2}((y + v) + (-y + v)) \in \overline{Z}.$$

This shows that $B[0, r] \subseteq \overline{Z}$, as desired. $\blacksquare$

3.4. The open mapping theorem

Lemma 3.4.4. *Let X be a Banach space, Y be a normed space (both over $\mathbb{K}$) and $T \in \mathcal{B}(X, Y)$. Assume there exists some $r > 0$ such that*

$$B[0, r] \subseteq \overline{T(B(0, 1))}.$$

Let $0 < t < 1$. Then we have

$$B[0, (1 - t)r] \subseteq T(B(0, 1)).$$

Proof. It suffices to show that $B[0, r] \subseteq T(B(0, (1 - t)^{-1}))$.

So let $y \in B[0, r]$, and set $Z := T(B(0, 1)) \subseteq Y$. Then $y \in \overline{Z}$ (using the assumption). Choose ε such that $0 < \varepsilon < t < 1$ and pick $y_1 \in Z$ such that $\|y - y_1\| < \varepsilon r$.

Note that

$$\left\| \frac{1}{\varepsilon}(y - y_1) \right\| < r, \text{ so } \frac{1}{\varepsilon}(y - y_1) \in B[0, r] \subseteq \overline{Z}.$$

Thus we can pick $y'_2 \in Z$ such that

$$\left\| \frac{1}{\varepsilon}(y - y_1) - y'_2 \right\| < \varepsilon r.$$

Set $y_2 := \varepsilon y'_2 \in \varepsilon Z$. Then we have $\|y - (y_1 + y_2)\| < \varepsilon^2 r$.

Proceeding inductively, we can construct a sequence $\{y_n\}_{n \in \mathbb{N}}$ such that $y_n \in \varepsilon^{n-1} Z$ and

$$\left\| y - \sum_{k=1}^n y_k \right\| < \varepsilon^n r$$

for every $n \in \mathbb{N}$. Since $\varepsilon^n r \rightarrow 0$ as $n \rightarrow \infty$ we have $y = \sum_{k=1}^{\infty} y_k$.

For $n \in \mathbb{N}$, note that

$$\varepsilon^{n-1} Z = \varepsilon^{n-1} T(B(0, 1)) = T(B(0, \varepsilon^{n-1})).$$

Thus, $y_n \in T(B(0, \varepsilon^{n-1}))$. So there exists $x_n \in X$ such that $\|x_n\| < \varepsilon^{n-1}$ and $y_n = T(x_n)$. We then have

$$\sum_{n=1}^{\infty} \|x_n\| \leq \sum_{n=1}^{\infty} \varepsilon^{n-1} = \frac{1}{1 - \varepsilon} < \infty.$$

Since X is Banach space, this implies that the series $\sum_{n=1}^{\infty} x_n$ converges to some $x \in X$ satisfying $\|x\| \leq (1 - \varepsilon)^{-1}$. Now, as T is continuous, we get that

$$T(x) = \sum_{n=1}^{\infty} T(x_n) = \sum_{n=1}^{\infty} y_n = y.$$

This shows that $y \in T(B[0, (1 - \varepsilon)^{-1}])$.

Since $(1 - \varepsilon)^{-1} < (1 - t)^{-1}$, we have $B[0, (1 - \varepsilon)^{-1}] \subseteq B(0, (1 - t)^{-1})$. Hence we get that $y \in T(B(0, (1 - t)^{-1}))$, as desired. $\blacksquare$

3. On Banach spaces and bounded linear operators

We are now ready to prove the open mapping theorem (Theorem 3.4.1).

Proof of the open mapping theorem. By Lemma 3.4.2, it suffices to show that there exists $\delta > 0$ such that $B[0, \delta] \subseteq T(B(0, 1))$.

Since T is surjective, we have

$$Y = T(X) = \bigcup_{n \in \mathbb{N}} T(B(0, n)) \subseteq \bigcup_{n \in \mathbb{N}} \overline{T(B(0, n))} \subseteq Y,$$

hence $Y = \bigcup_{n \in \mathbb{N}} \overline{T(B(0, n))}$.

Since Y is complete, Corollary 3.2.3 implies that there must exist some $n \in \mathbb{N}$ such that $\overline{T(B(0, n))}$ has nonempty interior. Since $\overline{T(B(0, n))} = n \overline{T(B(0, 1))}$, and $y \mapsto ny$ is a homeomorphism of Y , $\overline{T(B(0, 1))}$ has nonempty interior. So there exists some $y \in Y$ and $r > 0$ such that

$$B[y, r] \subseteq \overline{T(B(0, 1))}.$$

Lemma 3.4.3 gives that $B[0, r] \subseteq \overline{T(B(0, 1))}$. We can then pick $t \in (0, 1)$, set $\delta := (1 - t)r$ and use Lemma 3.4.4 to obtain that $B[0, \delta] \subseteq T(B(0, 1))$, as we wanted to show. ■

In the open mapping theorem, the assumption that X and Y are Banach spaces can not be removed. We will illustrate this in the exercises. The following corollary is sometimes called the *inverse operator theorem*.

Corollary 3.4.5. *Let X, Y be Banach spaces (over $\mathbb{K}$) and $T \in \mathcal{B}(X, Y)$. Assume T is bijective. Then $T^{-1} \in \mathcal{B}(Y, X)$.*

Proof. The open mapping theorem gives that T is open, hence that T^{-1} is continuous. Equivalently, T^{-1} is bounded. ■

We include some useful consequences of this result.

Corollary 3.4.6. *Let X, Y be Banach spaces (over $\mathbb{K}$) and $T \in \mathcal{B}(X, Y)$. Assume that T has closed range. Then $X/\ker(T)$ and $T(X)$ are Banach spaces, which are isomorphic as normed spaces.*

Proof. Since T is bounded, $\ker(T)$ is closed, so $X/\ker(T)$ is a Banach space (w.r.t. the quotient norm) by Theorem 3.1.4. Further, since $T(X)$ is a closed subspace of Y , which is complete, $T(X)$ is also complete, hence a Banach space. By (the proof of) Corollary 3.1.6, $\tilde{T} : X/\ker(T) \rightarrow T(X)$ is an isomorphism of vector spaces. Moreover, $\tilde{T}$ is bounded by Proposition 3.1.7. Finally, the inverse operator theorem gives that the inverse of $\tilde{T}$ is bounded. Thus, $\tilde{T}$ is an isomorphism of normed spaces. ■

3.5. The closed graph theorem

Example 3.4.7. Let Ω be a compact metric space and E be a closed subset of Ω . Let $T : (C(\Omega), \|\cdot\|_\infty) \rightarrow (C(E), \|\cdot\|_\infty)$ be the bounded linear map given by

$$T(f) = f|_E \quad (= \text{the restriction of } f \text{ to } E).$$

Then $\ker(T) = \{f \in C(\Omega) : f = 0 \text{ on } E\}$. Tietze's extension theorem (cf. your course in topology) gives that T is surjective. In particular, T has closed range. Thus, Corollary 3.4.6 applies, giving that the associated map $\tilde{T} : C(\Omega)/\ker(T) \rightarrow C(E)$ is an isomorphism of normed spaces.

It is easy to see that $\|T\| = 1$, so $\|\tilde{T}\| = 1$. It can actually be shown that $\tilde{T}$ is isometric, but we leave this as an exercise.

Remark 3.4.8. If we skip the assumption that T has closed range, then the conclusion of Corollary 3.4.6 does not necessarily hold.

Finally, we also mention:

Corollary 3.4.9. *Let X be a vector space (over $\mathbb{K}$) equipped with two norms $\|\cdot\|$ and $\|\cdot\|'$ such that both $(X, \|\cdot\|)$ and $(X, \|\cdot\|')$ are Banach spaces. Assume there exists $C > 0$ such that $\|x\|' \leq C\|x\|$ for all $x \in X$. Then $\|\cdot\|$ and $\|\cdot\|'$ are equivalent norms on X .*

Proof. The assumption gives that the identity operator I on X is bijective linear map from $(X, \|\cdot\|)$ into $(X, \|\cdot\|')$ which is bounded. By Corollary 3.4.5, $I = I^{-1}$ is a bounded linear map from $(X, \|\cdot\|')$ into $(X, \|\cdot\|)$. This means that there exists some $D > 0$ such that $\|x\| \leq D\|x\|'$ for all $x \in X$. Hence, $\|\cdot\|$ and $\|\cdot\|'$ are equivalent. ■

3.5 The closed graph theorem

Another fundamental result which ultimately relies on Baire's category theorem is the *closed graph theorem*. Before we can state it, we need some preparations.

Let X, Y be normed spaces over $\mathbb{K}$. The product space $X \times Y$ is then a vector space w.r.t. its natural operations $(x, y) + (x', y') = (x + x', y + y')$, $\lambda(x, y) = (\lambda x, \lambda y)$. We can equip $X \times Y$ with the norm given by

$$\|(x, y)\| := \|x\| + \|y\| \quad \text{for all } x \in X, y \in Y.$$

We leave it as an exercise to check that this is indeed a norm, and that $X \times Y$ becomes a Banach space if and only if X and Y are Banach spaces.

3. On Banach spaces and bounded linear operators

Consider now a map $T : X \rightarrow Y$. We recall that the *graph* of T is the subset of $X \times Y$ given by

$$G(T) = \{ (x, T(x)) : x \in X \}.$$

If T is linear, it is straightforward to check that $G(T)$ is a subspace of $X \times Y$. We say that T has *closed graph* when $G(T)$ is closed in $X \times Y$.

Lemma 3.5.1. *A map $T : X \rightarrow Y$ has closed graph if and only if the following condition holds:*

If $\{x_n\}$ is a sequence in X such that $\{x_n\}$ converges to $x \in X$ and $\{T(x_n)\}$ converges to $y \in Y$, then $y = T(x)$.

Proof. Assume T has closed graph and $\{x_n\}$ is a sequence in X such that $\{x_n\}$ converges to $x \in X$ and $\{T(x_n)\}$ converges to $y \in Y$. Then $\{(x_n, T(x_n))\}$ is a sequence in $G(T)$ converging to (x, y) since

$$\|(x_n, T(x_n)) - (x, y)\| = \|x_n - x\| + \|T(x_n) - y\| \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Since $G(T)$ is closed, we must have $(x, y) = (x', T(x'))$ for some $x' \in X$. Then $x = x'$ and $y = T(x')$, so $y = T(x)$. Thus the condition above holds.

Conversely, assume that this condition holds. To show that $G(T)$ is closed, it suffices to show that $\overline{G(T)} \subseteq G(T)$. So let $(x, y) \in \overline{G(T)}$. We can then find a sequence $\{x_n\}$ in X such that $\{(x_n, T(x_n))\}$ converges to (x, y) . Then we have

$$\|x_n - x\| \leq \|x_n - x\| + \|T(x_n) - y\| = \|(x_n, T(x_n)) - (x, y)\| \rightarrow 0 \text{ as } n \rightarrow \infty, \text{ and}$$

$$\|T(x_n) - y\| \leq \|x_n - x\| + \|T(x_n) - y\| = \|(x_n, T(x_n)) - (x, y)\| \rightarrow 0 \text{ as } n \rightarrow \infty,$$

that is, $\{x_n\}$ converges to x and $\{T(x_n)\}$ converges to y . Thus, using the assumption, we get $y = T(x)$. Hence, $(x, y) = (x, T(x)) \in G(T)$. This shows that $\overline{G(T)} \subseteq G(T)$, as we wanted to show. ■

It follows readily that if $T \in \mathcal{B}(X, Y)$, then T has closed graph. Indeed, if $\{x_n\}$ converges to $x \in X$ and $\{T(x_n)\}$ converges to $y \in Y$, then we have $y = \lim_{n \rightarrow \infty} T(x_n) = T(x)$ (since T is continuous). Conversely, we have:

Theorem 3.5.2 (The closed graph theorem). *Assume X and Y are Banach spaces, and $T : X \rightarrow Y$ is a linear map having closed graph. Then T is bounded.*

3.6. The second dual, reflexive spaces, and duals of quotient spaces

Proof. Let $P_1 : X \times Y \rightarrow X$ (resp. $P_2 : X \times Y \rightarrow Y$) denote the linear map defined by

$$P_1(x, y) = x \quad \text{resp.} \quad P_2(x, y) = y$$

for all $x, y \in X$. Then $\|P_1(x, y)\| = \|x\| \leq \|x\| + \|y\| = \|(x, y)\|$ for all $(x, y) \in X \times Y$, so P_1 is bounded. Similarly, P_2 is bounded.

Let now $P = P_1|_{G(T)}$ and $Q = P_2|_{G(T)}$ denote the restrictions of P_1 and P_2 to $G(T)$. Note that $G(T)$ is a Banach space since it is a closed subspace of $X \times Y$ (using the assumption on T). As the map P is obviously bijective, we can apply the inverse operator theorem to it and obtain that P has a bounded inverse $P^{-1} : X \rightarrow G(T)$. Since $P^{-1}(x) = (x, T(x))$ for all $x \in X$, we have that

$$T = Q P^{-1}.$$

As both Q and P^{-1} are bounded, T is bounded too. ■

A typical illustration of how the closed graph theorem may be used is the following result due to Hellinger and Toeplitz:

Theorem 3.5.3. *Assume H is a Hilbert space and $T : H \rightarrow H$ is a linear operator satisfying that*

$$\langle T(x), y \rangle = \langle x, T(y) \rangle \quad \text{for all } x, y \in H.$$

Then T is bounded (and self-adjoint).

Proof. Assume $\{x_n\}$ is a sequence in X such that it converges to $x \in X$ and $\{T(x_n)\}$ converges to $y \in Y$. Then for all $z \in X$ we have

$$\begin{aligned} \langle y, z \rangle &= \lim_{n \rightarrow \infty} \langle T(x_n), z \rangle = \lim_{n \rightarrow \infty} \langle x_n, T(z) \rangle \\ &= \langle x, T(z) \rangle = \langle T(x), z \rangle. \end{aligned}$$

This implies that $y = T(x)$. Hence T has closed graph, so it is bounded by Theorem 3.5.2. ■

3.6 The second dual, reflexive spaces, and duals of quotient spaces

Let X be a normed space (over $\mathbb{K}$). We recall that the *dual space* of X , $X^* = \mathcal{B}(X, \mathbb{K})$, is a Banach space w.r.t. the norm given by

$$\|\varphi\| = \sup\{|\varphi(x)| : x \in X, \|x\| \leq 1\}$$

3. On Banach spaces and bounded linear operators

for all $\varphi \in X^*$. We can also form the *second dual* (sometimes called the *bidual*) of X , namely $(X^*)^* = \mathcal{B}(X^*, \mathbb{K})$, which is also a Banach space w.r.t. the norm given by

$$\|F\| = \sup\{|F(\varphi)| : \varphi \in X^*, \|\varphi\| \leq 1\}$$

for all $F \in (X^*)^*$. We will usually write X^{**} instead of $(X^*)^*$.

Consider some $x \in X$. We can then define a linear functional $\hat{x} : X^* \rightarrow \mathbb{K}$ by

$$\hat{x}(\varphi) := \varphi(x) \quad \text{for all } \varphi \in X^*.$$

Since

$$|\hat{x}(\varphi)| = |\varphi(x)| \leq \|\varphi\| \|x\| \quad \text{for all } \varphi \in X^*,$$

we get that $\hat{x}$ is bounded, i.e., $\hat{x} \in X^{**}$, with $\|\hat{x}\| \leq \|x\|$. In fact, we have that

$$\|\hat{x}\| = \|x\|.$$

Indeed, by a corollary to the Hahn-Banach extension theorem (Corollary 1.1.14), we know that there exists $\varphi_x \in X^*$ such that $\|\varphi_x\| = 1$ and $\varphi_x(x) = \|x\|$. This gives that

$$|\hat{x}(\varphi_x)| = |\varphi_x(x)| = \|x\|,$$

hence that $\|\hat{x}\| \geq \|x\|$. Since $\|\hat{x}\| \leq \|x\|$, the claimed equality follows.

Now, define a map $J : X \rightarrow X^{**}$ by

$$J(x) := \hat{x} \quad \text{for all } x \in X.$$

Note that J is linear: for $x, y \in X$ and $\lambda \in \mathbb{K}$, we have

$$\widehat{x + \lambda y}(\varphi) = \varphi(x + \lambda y) = \varphi(x) + \lambda \varphi(y) = (\hat{x} + \lambda \hat{y})(\varphi)$$

for all $\varphi \in X^*$, i.e., $J(x + \lambda y) = J(x) + \lambda J(y)$. Moreover, J is isometric, since $\|J(x)\| = \|\hat{x}\| = \|x\|$ for all $x \in X$. Summing up, we have shown:

Proposition 3.6.1. *The map $J : X \rightarrow X^{**}$ is linear and isometric. We call it the canonical embedding of X into X^{**} .*

Remark 3.6.2. Set $\widetilde{X} := \overline{J(X)} \subseteq X^{**}$. Then $\widetilde{X}$ is a closed subspace of X^{**} , hence a Banach space. Moreover, the map $J : X \rightarrow \widetilde{X}$ is linear, isometric, and $J(X)$ is dense in $\widetilde{X}$. This means that $(\widetilde{X}, J)$ is a completion of X (in the sense of Remark 1.1.6 and Exercise 1.5).

Note also that if X is a Banach space, then $\widetilde{X} = J(X)$, i.e., $J(X)$ is closed in X^{**} . Indeed, as J is isometric and X is complete, $J(X)$ is complete too, and therefore closed.

3.6. The second dual, reflexive spaces, and duals of quotient spaces

Definition 3.6.3. We say that a Banach space X is *reflexive* when the canonical embedding $J : X \rightarrow X^{**}$ is surjective, i.e., $J(X) = X^{**}$.

Clearly, if X is a reflexive Banach space, then X is isometrically isomorphic to X^{**} via J . However, note that a Banach space X such that X is isometrically isomorphic to X^{**} is not necessarily reflexive, i.e., it may then happen that $J(X) \neq X^{**}$.

Example 3.6.4. Assume X is a finite-dimensional normed space. Then X is reflexive. Indeed, we then know that X^* is finite dimensional, with $\dim(X^*) = \dim(X)$. Hence, X^{**} is finite-dimensional too, with $\dim(X^{**}) = \dim(X^*)$. Thus we have

$$\dim(X^{**}) = \dim(X) = \dim(J(X)),$$

and this implies that $J(X) = X^{**}$.

Example 3.6.5. Assume H is Hilbert space. Then H is reflexive. Since $H^* = H$ as a set, this might seem obvious. But one has to be a little precautionous to show this.

For each $y \in H$, let $y^* \in H^*$ be given by $y^*(x) = \langle x, y \rangle$ for all $x \in H$. Recall that the Riesz representation theorem says that the map $y \mapsto y^*$ is a conjugate-linear bijective isometry from H onto H^* . Now, it is readily verified that H^* becomes a Hilbert space w.r.t. the inner product given by

$$\langle y^*, z^* \rangle_{H^*} := \langle z, y \rangle \quad \text{for all } y, z \in H.$$

For $x^* \in H^*$, let $x^{**} \in H^{**}$ be given by $x^{**}(y^*) := \langle y^*, x^* \rangle_{H^*}$ for all $y^* \in H^*$. Applying again the Riesz representation, we get that the map $x^* \mapsto x^{**}$ is a conjugate-linear bijective isometry from H^* onto H^{**} .

Now, the canonical embedding $J : H \rightarrow H^{**}$ is given by

$$[J(x)](y^*) = y^*(x) = \langle x, y \rangle \quad \text{for all } x, y \in H.$$

We want to show that J is surjective. So let $F \in H^{**}$. We may then write $F = x^{**}$ for some $x^* \in H^*$. Then, for all $y^* \in H^*$, we have

$$F(y^*) = x^{**}(y^*) = \langle y^*, x^* \rangle_{H^*} = \langle x, y \rangle = [J(x)](y^*).$$

Hence, $F = J(x)$. This shows that $J(H) = H^{**}$, as desired.

Example 3.6.6. Let $(\Omega, \mathcal{B}, \mu)$ be a measure space and $1 < p < \infty$. Consider the Banach space $(L^p, \|\cdot\|_p)$ where $L^p := L^p(\Omega, \mathcal{B}, \mu)$. Then L^p is reflexive.

3. On Banach spaces and bounded linear operators

To show this, let $1 < q < \infty$ be the conjugate exponent of p , i.e., q satisfies that $\frac{1}{p} + \frac{1}{q} = 1$. We may then also consider the Banach space $(L^q, \|\cdot\|_q)$. We recall that L^q is isometrically isomorphic to $(L^p)^*$ via the map $\Psi_p : L^q \rightarrow (L^p)^*$ defined by

$$[\Psi_p(g)](f) = \int_{\Omega} f g \, d\mu \quad \text{for all } f \in L^p, g \in L^q.$$

Similarly, the map $\Psi_q : L^p \rightarrow (L^q)^*$ given by

$$[\Psi_q(f)](g) = \int_{\Omega} g f \, d\mu \quad \text{for all } f \in L^p, g \in L^q,$$

is an isometric isomorphism. We note that this implies that

$$[\Psi_p(g)](f) = [\Psi_q(f)](g) \quad \text{for all } f \in L^p, g \in L^q. \quad (3.6.1)$$

Let now $F \in (L^p)^{**}$, i.e., $F : (L^p)^* \rightarrow \mathbb{C}$ is linear and bounded. To show that $J : L^p \rightarrow (L^p)^{**}$ is surjective, we have to show that $F = J(f)$ for some $f \in L^p$.

Observe first that $F \circ \Psi_p : L^q \rightarrow \mathbb{C}$ is then linear and bounded, that is, $F \circ \Psi_p \in (L^q)^*$. Thus, there exists some $f \in L^p$ such that

$$F \circ \Psi_p = \Psi_q(f).$$

Let's check that f does the job. Let $\varphi \in (L^p)^*$. We may then write $\varphi = \Psi_p(g)$ for some $g \in L^q$. Using (3.6.1), we get

$$\begin{aligned} F(\varphi) &= F(\Psi_p(g)) = (F \circ \Psi_p)(g) = [\Psi_q(f)](g) \\ &= [\Psi_p(g)](f) = \varphi(f) \\ &= [J(f)](\varphi). \end{aligned}$$

This shows that $F = J(f)$, as wanted.

As a particular case of this example, we note that $\ell^p(\mathbb{N})$ is reflexive for every $1 < p < \infty$. We will explain later how to see that $\ell^1(\mathbb{N})$ and $\ell^\infty(\mathbb{N})$ are not reflexive.

In order to prove that reflexivity is inherited by closed subspaces, we will need the following consequence of the Hahn-Banach extension theorem.

Proposition 3.6.7. *Let M be a closed subspace of a normed space X (over $\mathbb{K}$) which is proper, i.e., $M \neq X$. Let $x \in X \setminus M$ and set*

$$d := \inf\{\|x - m\| : m \in M\}.$$

Then $d > 0$ and there exists $\varphi \in X^*$ such that $\|\varphi\| = 1$, $\varphi = 0$ on M , and $\varphi(x) = d > 0$.

3.6. The second dual, reflexive spaces, and duals of quotient spaces

Proof. If it was the case that $d = 0$, this would imply that $x \in \overline{M}$, i.e., $x \in M$ (since M is closed), contradicting that $x \notin M$. Hence $d > 0$.

Set $X_0 := M + \mathbb{K}x := \{m + \lambda x \mid m \in M, \lambda \in \mathbb{K}\}$. Then X_0 is a subspace of X and every element of X_0 is of the form $m + \lambda x$ for some uniquely determined $m \in M$ and $\lambda \in \mathbb{K}$. Thus we can define a linear functional $\varphi_0 : X_0 \rightarrow \mathbb{K}$ by

$$\varphi_0(m + \lambda x) := \lambda d \quad \text{for all } m \in M, \lambda \in \mathbb{K}.$$

Then φ_0 is bounded with $\|\varphi_0\| \leq 1$:

Indeed, assume $y = m + \lambda x$, for some $m \in M$ and $\lambda \in \mathbb{K}$. If $\lambda \neq 0$, then $(-1/\lambda)m \in M$, so we have

$$|\varphi_0(y)| = |\varphi_0(m + \lambda x)| = |\lambda| d \leq |\lambda| \|x - (-1/\lambda)m\| = \|m + \lambda x\| = \|y\|.$$

On the other hand, if $\lambda = 0$, then $|\varphi_0(y)| = |\varphi_0(m)| = 0 \leq \|m\| = \|y\|$. Thus, $|\varphi_0(y)| \leq \|y\|$ for all $y \in X_0$, proving the claim above.

Moreover, we also have $\|\varphi_0\| \geq 1$, hence $\|\varphi_0\| = 1$:

Indeed, let $n \in \mathbb{N}$. Then $d < d(1 + 1/n)$, so we can pick $m_n \in M$ such that $\|x - m_n\| < d(1 + 1/n)$. Since $\|x - m_n\| \neq 0$, we get

$$\left| \varphi_0\left(\frac{1}{\|x - m_n\|}(x - m_n)\right) \right| = \frac{1}{\|x - m_n\|} d > \frac{1}{(1 + 1/n)} = \frac{n}{n + 1}.$$

This implies that $\|\varphi_0\| \geq \frac{n}{n+1}$. Letting $n \rightarrow \infty$, we get that $\|\varphi_0\| \geq 1$, as asserted above.

Applying the Hahn-Banach extension theorem, we can extend φ_0 to some $\varphi \in X^*$ such that $\|\varphi\| = \|\varphi_0\|$, hence satisfying $\|\varphi\| = 1$. We then have $\varphi(m) = \varphi_0(m) = 0$ for all $m \in M$, and $\varphi(x) = \varphi_0(x) = d$, as desired. ■

Remark 3.6.8. Proposition 3.6.7 can be used to show that if X is a Banach space such that X^* is separable, then X is separable too.

Theorem 3.6.9. *Let M be a closed subspace of a reflexive Banach space X . Then M is reflexive.*

Proof. By assumption, the canonical embedding $J_X : X \rightarrow X^{**}$ is surjective. We have to show that the canonical embedding $J_M : M \rightarrow M^{**}$ is also surjective.

Consider $F \in M^{**}$. Let $R : X^* \rightarrow M^*$ denote the restriction map, i.e., $R(\varphi) = \varphi|_M$ for all $\varphi \in X^*$. One readily checks that R is linear and bounded. So $F \circ R \in X^{**}$. We may therefore define $y \in X$ by

$$y := (J_X)^{-1}(F \circ R), \text{ i.e., } J_X(y) = F \circ R.$$

3. On Banach spaces and bounded linear operators

We claim that $y \in M$ and $J_M(y) = F$. (This will show that J_M is surjective and finish the proof.)

Assume (for contradiction) that $y \notin M$. Applying Proposition 3.6.7, we can then find some $\varphi \in X^*$ such that $\varphi = 0$ on M , i.e., $R(\varphi) = 0$, and $\varphi(y) \neq 0$. But this gives that

$$\varphi(y) = [J_X(y)](\varphi) = (F \circ R)(\varphi) = F(R(\varphi)) = F(0) = 0,$$

hence a contradiction. So we must have $y \in M$.

Moreover, consider $\psi \in M^*$. Let $\tilde{\psi} \in X^*$ be any extension of ψ (which exists by the Hahn-Banach extension theorem). Then $R(\tilde{\psi}) = \psi$, so

$$[J_M(y)](\psi) = \psi(y) = \tilde{\psi}(y) = [J_X(y)](\tilde{\psi}) = (F \circ R)(\tilde{\psi}) = F(R(\tilde{\psi})) = F(\psi).$$

This shows that $F = J_M(y)$, as desired. ■

It is now appropriate to briefly introduce the adjoint of a bounded linear map between two Banach spaces.

Proposition 3.6.10. *Let X, Y be Banach spaces (over $\mathbb{K}$) and $T \in \mathcal{B}(X, Y)$.*

- a) *Let $\psi \in Y^*$. Then $\psi \circ T \in X^*$, with $\|\psi \circ T\| \leq \|\psi\| \|T\|$.*
- b) *Using a), we may define a map $T^* : Y^* \rightarrow X^*$ by*

$$T^*(\psi) := \psi \circ T \quad \text{for all } \psi \in Y^*.$$

Then $T^* \in \mathcal{B}(Y^*, X^*)$, with $\|T^*\| = \|T\|$.

The map T^* is called the Banach space adjoint of T .²

Proof. a) The map $\psi \circ T : X \rightarrow \mathbb{K}$, being the composition of two linear bounded maps, is linear and bounded (cf. Proposition 1.1.18), with $\|\psi \circ T\| \leq \|\psi\| \|T\|$.

- b) It is straightforward to check that T^* is linear. As

$$\|T^*(\psi)\| = \|\psi \circ T\| \leq \|\psi\| \|T\|$$

for every $\psi \in Y^*$ by a), T^* is bounded, with $\|T^*\| \leq \|T\|$. To show that $\|T\| \leq \|T^*\|$, hence that $\|T^*\| = \|T\|$, let $x \in X$ and set $y := T(x) \in Y$.

²When $\mathbb{K} = \mathbb{C}$, and X and Y are complex Hilbert spaces, then T^* is not exactly the same as the Hilbert space adjoint of T . As we are not going to make use of Banach space adjoints outside the present section, we don't discuss this any further.

3.6. The second dual, reflexive spaces, and duals of quotient spaces

Then it follows from Proposition 1.1.14 that we can find some $\psi \in Y^*$ such that $\|\psi\| \leq 1$ and $\psi(y) = \|y\|$, i.e., $\psi(T(x)) = \|T(x)\|$. Thus,

$$\begin{aligned} \|T(x)\| &= |\psi(T(x))| = |(\psi \circ T)(x)| \\ &= |[T^*(\psi)](x)| \leq \|T^*(\psi)\| \|x\| \\ &\leq \|T^*\| \|\psi\| \|x\| \leq \|T^*\| \|x\|. \end{aligned}$$

This shows that $\|T\| \leq \|T^*\|$, as desired. ■

Our next result is intuitively quite obvious.

Proposition 3.6.11. *Let X, Y be Banach spaces (over $\mathbb{K}$) and assume X is isometrically isomorphic to Y via a map $T : X \rightarrow Y$. Then Y^* is isometrically isomorphic to X^* via the map $T^* : Y^* \rightarrow X^*$. It follows that X^{**} is isometrically isomorphic to Y^{**} . Moreover, X is reflexive if and only if Y is reflexive.*

We leave the proof of this proposition as Exercise 3.24. Another remarkable property of reflexivity is the following:

Theorem 3.6.12. *Let X be a Banach space (over $\mathbb{K}$). Then X is reflexive if and only if X^* is reflexive.*

Proof. We will denote the triple dual of X , namely $((X^*)^*)^*$, by X^{***} . Note that $X^{***} = (X^*)^{**} = (X^{**})^*$. Let $J_X : X \rightarrow X^{**}$ denote the canonical embedding of X into X^{**} and $J_{X^*} : X^* \rightarrow X^{***}$ denote the canonical embedding of X^* into X^{***} .

Assume that X is reflexive, i.e., J_X is surjective. We have to show that J_{X^*} is surjective. So let $\Phi \in X^{***}$. Define $\varphi \in X^*$ by $\varphi := \Phi \circ J_X$ and consider $F \in X^{**}$. Since $F = J_X(x)$ for some $x \in X$, we get

$$[J_{X^*}(\varphi)](F) = F(\varphi) = [J_X(x)](\varphi) = \varphi(x) = (\Phi \circ J_X)(x) = \Phi(F).$$

Thus, $J_{X^*}(\varphi) = \Phi$. Hence, J_{X^*} is surjective.

Conversely, assume that X^* is reflexive. Using the implication we have just proven, we get that $X^{**} = (X^*)^*$ is reflexive. Now, $J_X(X)$ is a closed subspace of X^{**} , so it follows from Theorem 3.6.9 that $J_X(X)$ is reflexive. Since J_X is an isometric isomorphism from X onto $J_X(X)$, we readily get that X is reflexive (cf. Proposition 3.6.11). ■

Example 3.6.13. Consider $c_0(\mathbb{N}) := \{f \in \ell^\infty(\mathbb{N}) : \lim_{n \rightarrow \infty} f(n) = 0\}$, which is a closed subspace of $\ell^\infty(\mathbb{N})$, hence a Banach space. We will sketch

3. On Banach spaces and bounded linear operators

how it can be shown that the canonical embedding $J : c_0(\mathbb{N}) \rightarrow c_0(\mathbb{N})^{**}$ is not surjective, hence that $c_0(\mathbb{N})$ is not reflexive.

Note that it will then follow from Theorem 3.6.9 that $\ell^\infty(\mathbb{N})$ is not reflexive. Moreover, using Theorem 3.6.12, this will in turn imply that $\ell^1(\mathbb{N})$ is not reflexive either (since we know that $\ell^\infty(\mathbb{N})$ is isometrically isomorphic to $\ell^1(\mathbb{N})^*$).

As a first step we observe that $c_0(\mathbb{N})^*$ is isometrically isomorphic to $\ell^1(\mathbb{N})$ via the map $T : c_0(\mathbb{N})^* \rightarrow \ell^1(\mathbb{N})$ given by

$$[T(\varphi)](n) = \varphi(\delta_n) \quad \text{for all } \varphi \in c_0(\mathbb{N})^* \text{ and } n \in \mathbb{N},$$

where $\delta_n \in c_0(\mathbb{N})$ is given for each $n \in \mathbb{N}$ by

$$\delta_n(k) = \begin{cases} 1 & \text{if } k = n, \\ 0 & \text{if } k \neq n, \end{cases} \quad (k \in \mathbb{N}).$$

We leave it as an exercise to check this fact. Next, using Proposition 3.6.11, we get that $T^* : \ell^1(\mathbb{N})^* \rightarrow c_0(\mathbb{N})^{**}$ is an isometric isomorphism. Letting $S : \ell^\infty(\mathbb{N}) \rightarrow \ell^1(\mathbb{N})^*$ denote the (isometric) isomorphism given by

$$[S(f)](g) = \sum_{n=1}^{\infty} f(n)g(n) \quad \text{for all } f \in \ell^\infty(\mathbb{N}) \text{ and } g \in \ell^1(\mathbb{N}),$$

we get that $T^* \circ S : \ell^\infty(\mathbb{N}) \rightarrow c_0(\mathbb{N})^{**}$ is an (isometric) isomorphism.

Now, let $f \in c_0(\mathbb{N})$ and $\varphi \in c_0(\mathbb{N})^*$. Then we have

$$\begin{aligned} [(T^* \circ S)(f)](\varphi) &= [T^*(S(f))](\varphi) = [S(f) \circ T](\varphi) \\ &= [S(f)](T(\varphi)) = \sum_{n=1}^{\infty} f(n) [T(\varphi)](n) \\ &= \sum_{n=1}^{\infty} f(n) \varphi(\delta_n). \end{aligned}$$

On the other hand, since $f = \sum_{n=1}^{\infty} f(n)\delta_n$ (w.r.t. $\|\cdot\|_\infty$) (check!), we also have

$$[J(f)](\varphi) = \varphi(f) = \varphi\left(\sum_{n=1}^{\infty} f(n)\delta_n\right) = \sum_{n=1}^{\infty} f(n) \varphi(\delta_n).$$

Thus, $J(f) = (T^* \circ S)(f)$ for all $f \in c_0(\mathbb{N})$. Hence, we get

$$c_0(\mathbb{N})^{**} = (T^* \circ S)(\ell^\infty(\mathbb{N})) = J(c_0(\mathbb{N})) \cup (T^* \circ S)(\ell^\infty(\mathbb{N}) \setminus c_0(\mathbb{N})) \quad (\text{disjoint union}).$$

Since $\ell^\infty(\mathbb{N}) \setminus c_0(\mathbb{N}) \neq \emptyset$, this shows that $J(c_0(\mathbb{N})) \neq c_0(\mathbb{N})^{**}$, as desired.

3.6. The second dual, reflexive spaces, and duals of quotient spaces

We end this section with the following theorem, which illustrates how Banach space adjoints are useful to identify the dual of a quotient space and of a closed subspace (cf. Remark 3.1.8).

Theorem 3.6.14. *Let M be a closed subspace of a normed space X (over $\mathbb{K}$). Let $Q : X \rightarrow X/M$ denote the quotient map and $I_M : M \rightarrow X$ the inclusion map. Then we have:*

a) *The map $Q^* \in \mathcal{B}((X/M)^*, X^*)$ is isometric and its range is the annihilator $M^\perp = \{\varphi \in X^* : \varphi|_M = 0\}$ of M . Hence $(X/M)^*$ is isometrically isomorphic to $M^\perp$.*

b) *Let $\tilde{Q} : X^* \rightarrow X^*/M^\perp$ denote the quotient map. Then there exists an isometric isomorphism $\sigma : M^* \rightarrow X^*/M^\perp$ such that $\tilde{Q} = \sigma \circ (I_M)^*$.*

Proof. a) We show that $Q^*((X/M)^*) = M^\perp$ and leave the proof of the fact that Q^* is isometric as Exercise 3.26. By definition of Q^* , we have $Q^*(\psi) = \psi \circ Q$ for all $\psi \in (X/M)^*$. Thus, if $x \in M$, then for all $\psi \in (X/M)^*$, we have

$$[Q^*(\psi)](x) = \psi(Q(x)) = \psi(0 + M) = 0,$$

so $Q^*(\psi) \in M^\perp$. Hence, $Q^*((X/M)^*) \subseteq M^\perp$.

Conversely, let $\varphi \in M^\perp$. Then $M \subseteq \ker(\varphi)$ and $\varphi \in \mathcal{B}(X, \mathbb{K}) = X^*$. By Proposition 3.1.7 there exists $\tilde{\varphi} \in \mathcal{B}(X/M, \mathbb{K}) = (X/M)^*$ such that $\tilde{\varphi}(Q(x)) = \varphi(x)$ for all $x \in X$. Then

$$\varphi = \tilde{\varphi} \circ Q = Q^*(\tilde{\varphi}) \in Q^*((X/M)^*).$$

Hence, $M^\perp \subseteq Q^*((X/M)^*)$.

b) Let $\varphi_0 \in M^*$. By Corollary 1.1.13 we can extend φ_0 to some $\varphi \in X^*$. Define

$$\sigma(\varphi_0) := \tilde{Q}(\varphi) = \varphi + M^\perp \in X^*/M^\perp.$$

Note that if $\varphi' \in X^*$ is another extension of φ_0 , then for all $x \in M$ we have

$$(\varphi' - \varphi)(x) = \varphi'(x) - \varphi(x) = \varphi_0(x) - \varphi_0(x) = 0,$$

i.e., $\varphi' - \varphi \in M^\perp$, so we get that $\varphi' + M^\perp = \varphi + M^\perp$. This shows that the definition of $\sigma(\varphi_0)$ does not on the choice of extension of φ_0 . It is now easy to check that the associated map $\sigma : M^* \rightarrow X^*/M^\perp$ is linear. Further, we have:

• σ is surjective: Let $\eta \in X^*/M^\perp$. Then $\eta = \varphi + M^\perp$ for some $\varphi \in X^*$. Set $\varphi_0 = \varphi|_M \in M^*$. As φ is an extension of φ_0 , we get that

$$\sigma(\varphi_0) = \varphi + M^\perp = \eta.$$

3. On Banach spaces and bounded linear operators

- $\tilde{Q} = \sigma \circ (I_M)^*$: Let $\varphi \in X^*$. Then $(I_M)^*(\varphi) = \varphi|_M$, so, as above, we get that

$$(\sigma \circ (I_M)^*)(\varphi) = \sigma((I_M)^*(\varphi)) = \varphi + M^\perp = \tilde{Q}(\varphi).$$

- σ is isometric: Let $\varphi_0 \in M^*$. Then there exists some $\varphi \in X^*$ extending φ_0 and satisfying $\|\varphi\| = \|\varphi_0\|$, cf. Corollary 1.1.13. Then we have

$$\|\sigma(\varphi_0)\| = \|\varphi + M^\perp\| = \|\tilde{Q}(\varphi)\| \leq \|\varphi\| = \|\varphi_0\|.$$

On the other hand, consider $\psi \in M^\perp$. Then we have

$$\begin{aligned} \|\varphi + \psi\| &= \sup\{|(\varphi + \psi)(x)| : x \in X, \|x\| \leq 1\} \\ &\leq \sup\{|(\varphi + \psi)(x)| : x \in M, \|x\| \leq 1\} \\ &= \sup\{|\varphi_0(x)| : x \in M, \|x\| \leq 1\} = \|\varphi_0\|. \end{aligned}$$

Hence

$$\|\sigma(\varphi_0)\| = \|\varphi + M^\perp\| = \inf\{\|\varphi + \psi\| : \psi \in M^\perp\} \leq \|\varphi_0\|.$$

This shows that $\|\sigma(\varphi_0)\| = \|\varphi_0\|$. ■

3.7 Exercises

Exercise 3.1. Let X be a Banach space and assume that M is a finite-dimensional subspace of X .

Show that there exists a closed subspace of N of X such that X is the direct sum of M and N , i.e., $X = M + N$ and $M \cap N = \{0\}$. (A closed subspace M of X is called *complemented* when such a closed subspace N exists. This is not always the case.)

Exercise 3.2. In Example 3.1.1, check that the map $T : H/M \rightarrow M^\perp$ is well-defined, linear and bijective.

Exercise 3.3. Assume X is normed space and M is a closed subspace such that M is complete and X/M is complete (w.r.t. the quotient norm). Show that X is complete, i.e., X is a Banach space.

Exercise 3.4. Let (X, d) be a metric space.

a) Check that a subset N of X is nowhere dense if and only if its complement $X \setminus N$ contains a dense open subset of X .

b) A subset R of X is said to be *residual* if its complement is meagre. Show that R is residual if and only if it contains a countable intersection of dense open subsets of X .

c) Assume that (X, d) is complete. Use Baire's category theorem to show that every residual subset of X is dense in X .

Exercise 3.5. Set

$$R := \{f : [0, 1] \rightarrow \mathbb{R} : f \text{ is continuous and nowhere differentiable}\}.$$

Following Banach, prove that R is dense in the space $C_{\mathbb{R}}([0, 1])$ of real-valued continuous on $[0, 1]$ (w.r.t. $\|\cdot\|_{\infty}$) as follows:

a) Let $n \in \mathbb{N}$ and set

$$U_n := \left\{f \in C_{\mathbb{R}}([0, 1]) \mid \sup \left\{ \left| \frac{f(s) - f(t)}{s - t} \right| : s, t \in [0, 1], s \neq t \right\} > n \text{ for all } t \in [0, 1] \right\}.$$

Show that U_n is open and dense in $C_{\mathbb{R}}([0, 1])$.

b) Show that $\cap_{n=1}^{\infty} U_n \subseteq R$. Then use Exercise 3.4, parts b) and c), to deduce that R is dense in $C_{\mathbb{R}}([0, 1])$.

Exercise 3.6. Let X and Y be Banach spaces (both over $\mathbb{K}$). Assume $\{T_n\}_{n \in \mathbb{N}}$ is a sequence in $\mathcal{B}(X, Y)$ such that $\{T_n(x)\}_{n \in \mathbb{N}}$ is convergent for all x in a dense subspace of X . Show that $\{T_n(x)\}_{n \in \mathbb{N}}$ is convergent for all x in X if and only if $\sup\{\|T_n\| : n \in \mathbb{N}\} < \infty$.

Exercise 3.7. Consider the subspace X of $(\ell^{\infty}(\mathbb{N}), \|\cdot\|_{\infty})$ given by

$$X = \{f \in \ell^{\infty}(\mathbb{N}) : f(k) = 0 \text{ except for finitely many } k\text{'s}\}.$$

Define $S_n : X \rightarrow X$ by

$$[S_n(f)](k) = \begin{cases} kf(k) & \text{if } 1 \leq k \leq n, \\ 0 & \text{if } k > n \end{cases}$$

for all $f \in X$. Check that $\sup\{\|S_n(f)\|_{\infty} : n \in \mathbb{N}\} < \infty$ for every $f \in X$, while $\sup\{\|S_n\| : n \in \mathbb{N}\} = \infty$. This illustrates that the principle of uniform boundedness does not necessarily hold when X is not complete.

Further, let $S : X \rightarrow X$ be the linear map given by $[S(f)](k) = kf(k)$ for all $f \in X$ and $k \in \mathbb{N}$. Check that $\lim_{n \rightarrow \infty} S_n(f) = S(f)$ for all $f \in X$ and that S is unbounded. This illustrates that Corollary 3.3.2 does not necessarily hold when X is not complete.

Exercise 3.8. Let X, Y, Z be normed spaces (over $\mathbb{K}$). Let $B : X \times Y \rightarrow Z$ be a map which is bilinear and separately continuous, that is, for each $x_0 \in X$ (resp. $y_0 \in Y$), the map $y \mapsto B(x_0, y)$ (resp. $x \mapsto B(x, y_0)$) is a continuous linear map from Y (resp. X) into Z . Assume that X or Y is complete.

3. On Banach spaces and bounded linear operators

Show that B is bounded in the sense that there exists some $M > 0$ such that

$$\|B(x, y)\| \leq M \|x\| \|y\| \quad \text{for all } x, y \in X.$$

Deduce that B is continuous when we equip the vector space $X \times Y$ with the norm $\|(x, y)\| := \|x\| + \|y\|$.

Exercise 3.9. Let A be a nonempty subset of a normed space X over $\mathbb{K}$. Show that A is bounded in X if and only if $\{\varphi(x) : x \in A\}$ is a bounded subset of $\mathbb{K}$ for every $\varphi \in X^*$.

Exercise 3.10. Let X be a Banach space, Y be a normed space (both over $\mathbb{K}$) and assume $\{T_n\}_{n \in \mathbb{N}} \subseteq \mathcal{B}(X, Y)$ satisfies that $\{T_n(x)\}_{n \in \mathbb{N}}$ converges in Y for every x in X . Let $T \in \mathcal{B}(X, Y)$ be given by

$$T(x) = \lim_{n \rightarrow \infty} T_n(x) \text{ for every } x \in X.$$

The fact that T is necessarily bounded follows from Corollary 3.3.2.

a) Show that $\{T_n\}_{n \in \mathbb{N}}$ converges to T uniformly on compact subsets of X , that is, given any compact $K \subseteq X$ and $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $\|T_n(x) - T(x)\| < \varepsilon$ for all $x \in K$ and $n \geq N$.

b) Assume that $\{x_n\}_{n \in \mathbb{N}}$ is a sequence in X converging to some $x \in X$. Show that

$$T(x) = \lim_{n \rightarrow \infty} T_n(x_n).$$

Exercise 3.11. Let $f : \mathbb{N} \rightarrow \mathbb{C}$ be a function such that $\sum_{k=1}^{\infty} f(k)g(k) < \infty$ for every $g \in c_0(\mathbb{N})$. Show that $f \in \ell^1(\mathbb{N})$.

Exercise 3.12. Provide the missing details in Example 3.3.4.

Exercise 3.13. Let X be a Banach space and assume X_1 and X_2 are two closed subspaces of X such that X is the direct sum of X_1 and X_2 , that is, every $x \in X$ can be written in a unique way as $x = x_1 + x_2$ with $x_1 \in X_1$ and $x_2 \in X_2$. Show that there exists $M > 0$ such that

$$\|x_1\| + \|x_2\| \leq \|x_1 + x_2\| \quad \text{for all } x_1 \in X_1 \text{ and } x_2 \in X_2.$$

(*Hint.* Find a natural way to invoke the inverse operator theorem.)

Deduce that the naturally defined projection maps from X to X_1 and X_2 , respectively, are bounded.

Exercise 3.14. Assume that X is a Banach space. An operator $P \in \mathcal{B}(X)$ satisfying that $P^2 = P$ is called a *projection*.

Let $P \in \mathcal{B}(X)$ be a projection. Show that $P(X)$ is closed and $X = \ker(P) \oplus P(X)$.

Exercise 3.15. Consider again the subspace X of $(\ell^\infty(\mathbb{N}), \|\cdot\|_\infty)$ given by

$$X = \{f \in \ell^\infty(\mathbb{N}) : f(k) = 0 \text{ except for finitely many } k\text{'s}\},$$

as in Example 3.7. Define a linear operator $T : X \rightarrow X$ by

$$[T(f)](k) = \frac{1}{k} f(k) \quad \text{for } f \in X \text{ and } k \in \mathbb{N}.$$

Check that T is bounded and bijective, but T^{-1} is not bounded (so T is not open). This illustrates that the conclusion of the open mapping theorem does not necessarily hold when X and Y are not complete.

Exercise 3.16. Consider the identity map from $(C([0, 1]), \|\cdot\|_\infty)$ into $(C([0, 1]), \|\cdot\|_2)$, which is obviously a bijection. Check that it is bounded, but that its inverse is not (so it is not open). This illustrates that the conclusion of the open mapping theorem does not necessarily hold when Y is not complete, even if X is.

Exercise 3.17. A linear map $T : X \rightarrow Y$ between Banach spaces is said to be *bounded from below* (or bounded away from zero) if there exists some $\alpha > 0$ such that $\alpha \|x\| \leq \|T(x)\|$ for all $x \in X$ (cf. Exercise 4.18 in the case where $T \in \mathcal{B}(H)$).

- a) Check that T is bounded below whenever $T \in \mathcal{B}(X, Y)$ is invertible.
- b) Show that a linear map $T : X \rightarrow Y$ is *not* bounded from below if and only if there exists a sequence $\{x_n\}$ of unit vectors in X such that $\lim_{n \rightarrow \infty} T(x_n) = 0$.
- c) Assume $T \in \mathcal{B}(X, Y)$ is bounded from below. Show that T has closed range.

Exercise 3.18. Let H be a Hilbert space and $T \in \mathcal{B}(H)$. Show that the following conditions are equivalent:

- i) T is invertible in $\mathcal{B}(H)$.
- ii) T^* is invertible in $\mathcal{B}(H)$.
- iii) Both T and T^* are bounded from below (as defined in the previous exercise).
- iv) Both T and T^* are injective and T has closed range.
- v) T is bijective.

3. On Banach spaces and bounded linear operators

Note: The equivalence between i) and v) follows from the inverse operator theorem (Corollary 3.4.5).

Exercise 3.19. Show that the map $\tilde{T}$ in Example 3.4.7 is isometric.

Exercise 3.20. Let X, Y be Banach spaces (over $\mathbb{K}$) and $T : X \rightarrow Y$ be linear. Show that T is continuous if and only if the following condition holds:

If $\{x_n\}$ is a sequence in X which converges to 0 and $\{T(x_n)\}$ converges to some $y \in Y$, then $y = 0$.

Exercise 3.21. Let X and Y be normed spaces (over $\mathbb{K}$) and $T : X \rightarrow Y$ be linear. Assume that $\psi \circ T \in X^*$ for all $\psi \in Y^*$. Show that T is bounded.

Exercise 3.22. Set $X = C^1([0, 1])$, $Y = C([0, 1])$, and equip both spaces with the $\|\cdot\|_\infty$ -norm. Let $D : X \rightarrow Y$ be the linear operator given by $D(f) = f'$ (the derivative of f) for each $f \in X$. Check that D has closed graph and is unbounded. This illustrates that the conclusion of the closed graph theorem is not necessarily true when X is not complete, even if Y is.

Exercise 3.23. a) Let X be a reflexive Banach space. Show that every $\varphi \in X^*$ attains its norm on the closed unit ball of X .

Note. R.C. James showed that the converse statement also holds, i.e., if every $\varphi \in X^*$ attains its norm on the closed unit ball of X , then X is reflexive. However, the proof is hard and outside the scope of this course.

b) Consider the Banach space $(C([-\pi, \pi]), \|\cdot\|_\infty)$. Show that it is not reflexive.

(*Hint.* Study $\varphi \in C([-\pi, \pi])^*$ given by $\varphi(f) := \int_{-\pi}^{\pi} f(t) \sin(t) dt$ for all $f \in C([-\pi, \pi])$.)

Exercise 3.24. Prove that Proposition 3.6.11 holds. That is, let X, Y be Banach spaces (over $\mathbb{K}$) and assume that $T : X \rightarrow Y$ is an isometric isomorphism from X onto Y . Then show that Y^* is isometrically isomorphic to X^* via the map T^* . Moreover, show that X is reflexive if and only if Y is reflexive.

Exercise 3.25. Consider the Banach space

$$c_0(\mathbb{N}) := \{f \in \ell^\infty(\mathbb{N}) : \lim_{n \rightarrow \infty} f(n) = 0\},$$

as in Example 4.2.

a) Show that $c_0(\mathbb{N})^*$ is isometrically isomorphic to $\ell^1(\mathbb{N})$ via the map described in this example.

Alternatively, you may consider the inverse of this map, which maps each $g \in \ell^1(\mathbb{N})$ to $\varphi_g \in c_0(\mathbb{N})^*$, where $\varphi_g(f) := \sum_{n=1}^{\infty} f(n)g(n)$ for all $f \in c_0(\mathbb{N})$.

b) Using a), as done in Example 4.2, we get that $c_0(\mathbb{N})^{**}$ is isometrically isomorphic to $\ell^\infty(\mathbb{N})$. Another way to deduce that $c_0(\mathbb{N})$ is not reflexive is to use this fact to show that $c_0(\mathbb{N})$ is not isometrically isomorphic to $c_0(\mathbb{N})^{**}$. Do this by observing first that $c_0(\mathbb{N})$ is separable, while $\ell^\infty(\mathbb{N})$ is not.

Exercise 3.26. Show that the map $Q^* : (X/M)^* \rightarrow X^*$ in Theorem 3.6.14 a) is isometric.

Exercise 3.27. Set

$$c(\mathbb{N}) := \{f \in \ell^\infty(\mathbb{N}) : \lim_{n \rightarrow \infty} f(n) \text{ exists in } \mathbb{C}\}.$$

a) Show that $c(\mathbb{N})$ is a closed subspace of $\ell^\infty(\mathbb{N})$, hence is a Banach space.

b) Show that $c(\mathbb{N})^*$ is isometrically isomorphic to $\ell^1(\mathbb{N})$. Then show that $c(\mathbb{N})$ is not reflexive.

Exercise 3.28. Assume X is a Banach space having a direct sum decomposition $X = X_1 \oplus X_2$. Describe X^* in terms of X_1^* and X_2^* .

CHAPTER 4

On compact operators

4.1 Introduction to compact operators between normed spaces

In this section, X and Y will denote normed spaces, both over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, unless otherwise specified.

Definition 4.1.1. An operator $T \in \mathcal{L}(X, Y)$ is called *compact* if the sequence $\{T(x_n)\}_{n \in \mathbb{N}}$ has a convergent subsequence in Y whenever $\{x_n\}_{n \in \mathbb{N}}$ is a bounded sequence in X .

We set $\mathcal{K}(X, Y) := \{T \in \mathcal{L}(X, Y) : T \text{ is compact}\}$.

To appreciate this definition, the concept of relative compactness for subsets of a metric space will be helpful.

A subset A of a metric space (Z, d) is called *relatively compact* in Z if its closure $\bar{A}$ is a compact subset of Z . (Some authors say *precompact* instead of relatively compact.) Equivalently, and this may be taken as the definition for our purposes, a subset A of Z is relatively compact in Z if and only if every sequence in A has a subsequence which converges in Z . In comparison, we recall that A is compact if and only if every sequence in A has a subsequence which converges in A .

We also remark that a subset A of Z is bounded whenever A is relatively compact in Z : indeed, if A is not bounded, then we can pick (any) $z \in Z$ and find a sequence $\{a_n\}_{n \in \mathbb{N}}$ in A such that $d(a_n, z) > n$ for all $n \in \mathbb{N}$; it is then rather easy to see that $\{a_n\}_{n \in \mathbb{N}}$ has no convergent subsequence in Z , so A is not relatively compact.

Proposition 4.1.2. Let $T \in \mathcal{L}(X, Y)$. Then T is compact if and only if $T(B)$ is relatively compact in Y whenever B is a bounded subset of X .

4. On compact operators

Proof. Assume first that T is compact and let $B \subseteq X$ be bounded. We want to show that $T(B)$ is relatively compact in Y . So let $\{y_n\}_{n \in \mathbb{N}}$ be a sequence in $T(B)$. For each $n \in \mathbb{N}$ we may then write $y_n = T(x_n)$ for some $x_n \in B$. As the sequence $\{x_n\}_{n \in \mathbb{N}}$ lies in B , it is bounded. Hence, by compactness of T , $\{y_n\}_{n \in \mathbb{N}} = \{T(x_n)\}_{n \in \mathbb{N}}$ has a convergence subsequence in Y . Thus, $T(B)$ is relatively compact, as desired.

Conversely, assume that T maps bounded subsets of X into relatively compact subsets of Y . We want to show that T is compact. So let $\{x_n\}_{n \in \mathbb{N}}$ be a bounded sequence in X . Set $B := \{x_n : n \in \mathbb{N}\}$. Then B is a bounded subset of X , so $T(B) = \{T(x_n) : n \in \mathbb{N}\}$ is relatively compact in Y . As $\{T(x_n)\}_{n \in \mathbb{N}}$ is a sequence in $T(B)$, we can conclude that it has a convergent subsequence in Y . Thus, T is compact, as desired. ■

Corollary 4.1.3. *Assume $T \in \mathcal{L}(X, Y)$ is compact. Then T is bounded. Thus, $\mathcal{K}(X, Y) \subseteq \mathcal{B}(X, Y)$.*

Proof. Let B denote the closed unit ball in X (i.e., $B = X_1$). Since B is a bounded subset of X , we get from Proposition 4.1.2 that $T(B)$ is relatively compact subset of Y . This implies that $T(B)$ is bounded. Hence we can find $M > 0$ such that $\|T(x)\| \leq M$ for all $x \in B$, and it follows that T is bounded with $\|T\| \leq M$. ■

An important class of compact operators consists of the finite-rank operators in $\mathcal{B}(X, Y)$. We recall that $T \in \mathcal{L}(X, Y)$ is said to have finite-rank if $T(X)$ is finite-dimensional.

Proposition 4.1.4. *Assume that $T \in \mathcal{B}(X, Y)$ has finite-rank. Then T is compact.*

Proof. Assume $\{x_n\}_{n \in \mathbb{N}}$ is a bounded sequence in X , and let $M > 0$ be such that $\|x_n\| \leq M$ for all $n \in \mathbb{N}$. Then we have

$$\|T(x_n)\| \leq \|T\| \|x_n\| \leq \|T\| M$$

for all $n \in \mathbb{N}$. Now, the ball $B := \{y \in Y : \|y\| \leq \|T\| M\}$ is closed in Y . Considering $T(X)$ as a normed space w.r.t. to the norm it inherits from Y , we get that the set $K := T(X) \cap B$ is a closed and bounded subset of $T(X)$. Since $T(X)$ is finite-dimensional (by assumption), we get from Proposition 1.2.3 that K is compact in $T(X)$. As $\{T(x_n)\}_{n \in \mathbb{N}}$ is a sequence in K , we can therefore conclude that it has a convergent subsequence. This shows that T is compact. ■

4.1. Introduction to compact operators between normed spaces

Example 4.1.5. Consider $X = C([0, 1], \mathbb{R})$ (the continuous real-valued functions on $[0, 1]$) as a real Banach space with respect to the norm $\|\cdot\|_\infty$. For $g \in X$, define $T(g) : [0, 1] \rightarrow \mathbb{R}$ by

$$[T(g)](s) = \int_0^1 \sin(s-t) g(t) \, dt \quad \text{for all } s \in [0, 1].$$

Since $\sin(s-t) = \sin(s)\cos(t) - \cos(s)\sin(t)$, we have that

$$[T(g)](s) = \left(\int_0^1 \cos(t) g(t) \, dt \right) \sin(s) - \left(\int_0^1 \sin(t) g(t) \, dt \right) \cos(s)$$

for all $s \in [0, 1]$. It follows that $T(g) \in X$. Moreover, the map $T : X \rightarrow X$ sending g to $T(g)$ is clearly linear. As $T(X)$ is 2-dimensional, T has finite-rank. Further, since

$$|[T(g)](s)| \leq \int_0^1 |\sin(s-t)g(t)| \, dt \leq \int_0^1 |g(t)| \, dt \leq \|g\|_1$$

for all $s \in [0, 1]$, we get that $\|T(g)\|_\infty \leq \|g\|_1$ for all $g \in X$. Hence, T is bounded. We can therefore conclude that T is compact.

More generally, using the so-called Arzelà-Ascoli Theorem (cf. MAT2400 or [R, section 6.3]), it can be shown that if $K : [a, b] \times [c, d] \rightarrow \mathbb{R}$ is a continuous function, then the operator $T : C([c, d], \mathbb{R}) \rightarrow C([a, b], \mathbb{R})$ defined by

$$[T(g)](s) = \int_c^d K(s, t) g(t) \, dt \quad \text{for all } s \in [a, b],$$

is linear and compact. Similarly, if K takes complex values, then we may consider T as an operator from $C([c, d])$ into $C([a, b])$, and it again follows that T is compact. $\square$

Theorem 4.1.6. $\mathcal{K}(X, Y)$ is a subspace of $\mathcal{B}(X, Y)$. Moreover, if Y is a Banach space, then $\mathcal{K}(X, Y)$ is closed in $\mathcal{B}(X, Y)$ (w.r.t. operator norm), and it follows that $\mathcal{K}(X, Y)$ is a Banach space.

Proof. We leave the proof of the first assertion as an exercise. So assume that Y is Banach space, and let $\{T_n\}_{n \in \mathbb{N}}$ be a sequence in $\mathcal{K}(X, Y)$ converging to some $T \in \mathcal{B}(X, Y)$. To show that $\mathcal{K}(X, Y)$ is closed in $\mathcal{B}(X, Y)$, we have to show that T is compact.

So let $\{x_n\}_{n \in \mathbb{N}}$ be a bounded sequence in X . Choose $M > 0$ such that $\|x_n\| \leq M$ for all $n \in \mathbb{N}$.

- Since T_1 is compact, there exists a subsequence $\{x_{n_k}\}_{k \in \mathbb{N}}$ of $\{x_n\}_{n \in \mathbb{N}}$ such that $T_1(x_{n_k}) \rightarrow y_1$ as $k \rightarrow \infty$ for some $y_1 \in Y$.

We set $x_{1,k} := x_{n_k}$ for each $k \in \mathbb{N}$. We then have $T_1(x_{1,n}) \rightarrow y_1$ as $n \rightarrow \infty$.

4. On compact operators

- Similarly, since $\{x_{1,n}\}_{n \in \mathbb{N}}$ is bounded and T_2 is compact, we can find a sequence $\{x_{2,n}\}_{n \in \mathbb{N}}$, which is a subsequence of $\{x_{1,n}\}_{n \in \mathbb{N}}$, and therefore of $\{x_n\}_{n \in \mathbb{N}}$, such that $T_2(x_{2,n}) \rightarrow y_2$ as $n \rightarrow \infty$ for some $y_2 \in Y$.
- Proceeding inductively, for each $m \in \mathbb{N}$, $m \geq 2$, we can find a sequence $\{x_{m,n}\}_{n \in \mathbb{N}}$, which is a subsequence of $\{x_{m-1,n}\}_{n \in \mathbb{N}}$, and therefore of $\{x_n\}_{n \in \mathbb{N}}$, such that $T_m(x_{m,n}) \rightarrow y_m$ as $n \rightarrow \infty$ for some $y_m \in Y$.

We now set $x'_k := x_{k,k} \in X$ for each $k \in \mathbb{N}$, and claim that

$$\{T(x'_k)\}_{k \in \mathbb{N}} \text{ is a Cauchy sequence in } Y. \quad (4.1.1)$$

Since Y is complete, we will then be able to conclude that $\{T(x'_k)\}_{k \in \mathbb{N}}$ is convergent, hence that $\{T(x_n)\}_{n \in \mathbb{N}}$ has a convergent subsequence, which will show that T is compact.

To establish (4.1.1), we first observe that for any $k, l, m \in \mathbb{N}$, we have

$$\begin{aligned} \|T(x'_l) - T(x'_k)\| &\leq \|(T - T_m)(x'_l) + T_m(x'_l) - T_m(x'_k) + (T_m - T)(x'_k)\| \\ &\leq \|(T - T_m)(x'_l)\| + \|T_m(x'_l) - T_m(x'_k)\| + \|(T_m - T)(x'_k)\| \\ &\leq \|T - T_m\| \|x'_l\| + \|T_m(x'_l) - T_m(x'_k)\| + \|T_m - T\| \|x'_k\| \\ &\leq \|T_m(x'_l) - T_m(x'_k)\| + 2M \|T - T_m\|. \end{aligned}$$

Let then $\varepsilon > 0$ and choose $m \in \mathbb{N}$ such that $\|T - T_m\| < \varepsilon/3M$. By construction, for each $k \geq m$, we have that $T_m(x'_k) = T_m(x_{k,k})$ is an element of the sequence $\{T_m(x_{m,n})\}_{n \in \mathbb{N}}$, which is convergent to y_m . It follows that the sequence $\{T_m(x'_k)\}_{k \in \mathbb{N}}$ is convergent, hence that it is Cauchy. So we can pick $N \in \mathbb{N}$ such that $\|T_m(x'_l) - T_m(x'_k)\| < \varepsilon/3$ for all $k, l \geq N$. This gives that

$$\|T(x'_l) - T(x'_k)\| \leq \|T_m(x'_l) - T_m(x'_k)\| + 2M \|T - T_m\| < \varepsilon/3 + 2M (\varepsilon/3M) = \varepsilon$$

for all $k, l \geq N$. Hence we have shown that the claim (4.1.1) is true.

Finally, as Y is a Banach space, we know that $\mathcal{B}(X, Y)$ is a Banach space too, and this implies that $\mathcal{K}(X, Y)$, being closed in $\mathcal{B}(X, Y)$, is also a Banach space. ■

An immediate consequence is the following:

Corollary 4.1.7. *Assume Y is a Banach space and set*

$$\mathcal{F}(X, Y) := \{T \in \mathcal{B}(X, Y) : T \text{ has finite-rank}\}.$$

Then we have

$$\overline{\mathcal{F}(X, Y)} \subseteq \mathcal{K}(X, Y)$$

(the closure being taken w.r.t. operator norm).

4.1. Introduction to compact operators between normed spaces

Example 4.1.8. Let $1 \leq p < \infty$ and set $X := \ell^p(\mathbb{N})$, which we know is a Banach space w.r.t. $\|\cdot\|_p$. For each $\lambda \in \ell^\infty(\mathbb{N})$, we may consider the multiplication operator $M_\lambda \in \mathcal{B}(X)$ given by

$$[M_\lambda(x)](n) = \lambda(n)x(n)$$

for all $x \in X$ and all $n \in \mathbb{N}$. One readily checks that $\|M_\lambda\| = \|\lambda\|_\infty$.

Now, assume that $\lambda \in c_0(\mathbb{N})$, i.e., $\lim_{n \rightarrow \infty} \lambda(n) = 0$. Then M_λ is compact. Indeed, for each $k \in \mathbb{N}$, let $\lambda^{(k)} \in \ell^\infty(\mathbb{N})$ be defined by

$$\lambda^{(k)}(n) = \begin{cases} \lambda(n) & \text{if } 1 \leq n \leq k, \\ 0 & \text{otherwise,} \end{cases}$$

for every $n \in \mathbb{N}$. Then it is clear that each $M_{\lambda^{(k)}}$ has finite-rank; moreover,

$$\|M_\lambda - M_{\lambda^{(k)}}\| = \|\lambda - \lambda^{(k)}\|_\infty \rightarrow 0 \quad \text{as } k \rightarrow \infty.$$

Thus $M_\lambda \in \overline{\mathcal{F}(X, X)} \subseteq \mathcal{K}(X, X)$.

□

We set $\mathcal{K}(X) := \mathcal{K}(X, X)$, so that $\mathcal{K}(X)$ is a subspace of $\mathcal{B}(X)$. If X is finite-dimensional, then every operator in $\mathcal{B}(X)$ has finite-rank, so $\mathcal{K}(X) = \mathcal{B}(X)$. On the other hand, if X is infinite-dimensional, then $\mathcal{K}(X) \neq \mathcal{B}(X)$. Indeed, the identity operator I_X is not compact in this case, because we then know that the closed unit ball X_1 is bounded but not compact, so that I_X maps the bounded set X_1 onto itself, which is not compact.

We also mention (cf. Exercise 4.1) that $\mathcal{K}(X)$ is a *two-sided ideal* in $\mathcal{B}(X)$, meaning that we have

$$ST \in \mathcal{K}(X) \text{ if } S \in \mathcal{B}(X) \text{ and } T \in \mathcal{K}(X), \text{ or if } S \in \mathcal{K}(X) \text{ and } T \in \mathcal{B}(X).$$

This property implies that no operator in $\mathcal{K}(X)$ can have a bounded inverse when X is infinite-dimensional (because if some $T \in \mathcal{K}(X)$ had an inverse $T^{-1} \in \mathcal{B}(X)$, then we would have that $I_X = T^{-1}T \in \mathcal{K}(X)$, hence that $\dim(X) < \infty$).

We will come back later to other properties of compact operators on Banach spaces.

4.2 Compact operators on Hilbert spaces

In view of Corollary 4.1.7, it is natural to wonder whether any compact operator from a normed space to a Banach space may be approximated in operator norm by bounded finite-rank operators. This problem was open until 1973, when a counterexample was found by the Swedish mathematician P. Enflo. Happily, the situation is as nice as possible when the target space is a Hilbert space.

Theorem 4.2.1. *Let X be a normed space and H be a Hilbert space (both over $\mathbb{K}$). Then we have*

$$\overline{\mathcal{F}(X, H)} = \mathcal{K}(X, H).$$

Proof. By Corollary 4.1.7, we only have to show that $\mathcal{K}(X, H) \subseteq \overline{\mathcal{F}(X, H)}$. So let $T \in \mathcal{K}(X, H)$ and $\varepsilon > 0$. We need to prove that there exists $S \in \mathcal{F}(X, H)$ such that $\|T - S\| \leq \varepsilon$. Clearly, we can assume $T \neq 0$.

Set $A := \overline{T(X_1)}$. Since X_1 is bounded and T is compact, the set A is compact in H . As H is a metric space, this implies that A is totally bounded (cf. Proposition 3.5.12 in Lindström's book). Hence we can cover A with some open balls $B_1, \dots, B_n$ of radius $\varepsilon/4$, having respective centers $a_1, \dots, a_n \in A$. For each $j = 1, \dots, n$, we can then find $x_j \in X_1$ such that $\|a_j - T(x_j)\| < \varepsilon/4$.

Set now $F := \text{Span}(\{T(x_1), \dots, T(x_n)\})$, which is a finite dimensional subspace of H , and let P_F denote the orthogonal projection of H on F . Since the range of $P_F T$ is contained in F , $P_F T$ has finite-rank, so $P_F T \in \mathcal{F}(X, H)$. We claim that

$$\|T - P_F T\| \leq \varepsilon.$$

Indeed, let $x \in X_1$. Then $T(x) \in A$, so $T(x) \in B_j$ for some $j \in \{1, \dots, n\}$. Hence,

$$\|T(x) - T(x_j)\| \leq \|T(x) - a_j\| + \|a_j - T(x_j)\| < \varepsilon/4 + \varepsilon/4 = \varepsilon/2.$$

Further, since $T(x_j) \in F$, we have that $P_F(T(x_j)) = T(x_j)$. Thus, using also that $\|P_F\| = 1$, we obtain that

$$\begin{aligned} \|(T - P_F T)(x)\| &= \|T(x) - T(x_j) + (P_F T)(x_j) - (P_F T)(x)\| \\ &\leq \|T(x) - T(x_j)\| + \|P_F(T(x_j) - T(x))\| \\ &\leq \|T(x) - T(x_j)\| + \|P_F\| \|T(x_j) - T(x)\| \\ &= 2 \|T(x) - T(x_j)\| \\ &< 2 \cdot \varepsilon/2 = \varepsilon. \end{aligned}$$

4.2. Compact operators on Hilbert spaces

As this holds for every $x \in X_1$, the claim follows. Hence, setting $S := P_F T$, we are done. ■

Remark 4.2.2. Let X be a normed space and H be a Hilbert space, and let $T \in \mathcal{K}(X, H)$. Then it can be shown that $\overline{T(X)}$ is separable. We leave this as an exercise. □

Theorem 4.2.1 immediately gives:

Corollary 4.2.3. *Let H be a Hilbert space. Set $\mathcal{K}(H) := \mathcal{K}(H, H)$ and $\mathcal{F}(H) := \mathcal{F}(H, H)$. Then we have*

$$\overline{\mathcal{F}(H)} = \mathcal{K}(H).$$

An application of this result is the following:

Corollary 4.2.4. *Let H be a Hilbert space and let $T \in \mathcal{K}(H)$. Then $T^* \in \mathcal{K}(H)$. In other words, $\mathcal{K}(H)$ is closed under the adjoint operation.*

Proof. Using Corollary 4.2.3, we can find a sequence $\{T_n\}_{n \in \mathbb{N}}$ in $\mathcal{F}(H)$ such that $\|T - T_n\| \rightarrow 0$ as $n \rightarrow \infty$. Now, it is not difficult to see that $\mathcal{F}(H)$ is closed under the adjoint operation (cf. Exercise 2.20). Hence, $\{T_n^*\}_{n \in \mathbb{N}}$ is a sequence in $\mathcal{F}(H)$, and we have

$$\|T^* - T_n^*\| = \|(T - T_n)^*\| = \|T - T_n\| \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Thus, $T^* \in \overline{\mathcal{F}(H)} = \mathcal{K}(H)$. ■

An interesting class of compact operators on H containing $\mathcal{F}(H)$ consists of the so-called *Hilbert-Schmidt operators*. For simplicity, we consider for the moment only the case where H is separable and infinite-dimensional. We note that every orthonormal basis for H is then countably infinite: indeed, assume (for contradiction) that H had an uncountable orthonormal basis $\mathcal{B}$. Then, as $\|u - u'\| = \sqrt{2}$ for all distinct $u, u' \in \mathcal{B}$, it is not difficult to see that any dense subset of H would have to be uncountable, contradicting the separability of H .

Lemma 4.2.5. *Assume H is a separable, infinite-dimensional Hilbert space (over $\mathbb{K}$). Let $\mathcal{B} = \{u_j\}_{j \in \mathbb{N}}$ and $\mathcal{C} = \{v_j\}_{j \in \mathbb{N}}$ be orthonormal bases for H , and let $T \in \mathcal{B}(H)$. Then we have*

$$\sum_{j=1}^{\infty} \|T(u_j)\|^2 = \sum_{j=1}^{\infty} \|T(v_j)\|^2.$$

4. On compact operators

Proof. Using Parseval's identity (two times), we get

$$\begin{aligned} \sum_{j=1}^{\infty} \|T(u_j)\|^2 &= \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} |\langle T(u_j), v_k \rangle|^2 = \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} |\langle u_j, T^*(v_k) \rangle|^2 \\ &= \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} |\langle T^*(v_k), u_j \rangle|^2 = \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} |\langle T^*(v_k), u_j \rangle|^2 \\ &= \sum_{k=1}^{\infty} \|T^*(v_k)\|^2. \end{aligned}$$

Note that the change of order of summation at the second but last step above is allowed since we are dealing with sums of non-negative numbers. Applying what we have done to the case where $\mathcal{B} = \mathcal{C}$, i.e., $u_j = v_j$ for every $j \in \mathbb{N}$, we get that

$$\sum_{j=1}^{\infty} \|T(v_j)\|^2 = \sum_{k=1}^{\infty} \|T^*(v_k)\|^2.$$

Thus we obtain that

$$\sum_{j=1}^{\infty} \|T(u_j)\|^2 = \sum_{k=1}^{\infty} \|T^*(v_k)\|^2 = \sum_{j=1}^{\infty} \|T(v_j)\|^2,$$

as desired. ■

Remark 4.2.6. An analogous result is true when H is finite-dimensional and $\mathcal{B} = \{u_j\}_{1 \leq j \leq n}$, $\mathcal{C} = \{v_j\}_{1 \leq j \leq n}$ are orthonormal bases for H (where $n = \dim(H)$). □

Definition 4.2.7. Let H be a separable, infinite-dimensional Hilbert space (over $\mathbb{K}$). An operator $T \in \mathcal{B}(H)$ is called an *Hilbert-Schmidt operator* when we have

$$\sum_{j=1}^{\infty} \|T(u_j)\|^2 < \infty$$

for some orthonormal basis $\mathcal{B} = \{u_j\}_{j \in \mathbb{N}}$ of H , in which case we set

$$\|T\|_2 := \left(\sum_{j=1}^{\infty} \|T(u_j)\|^2 \right)^{1/2}.$$

Lemma 4.2.5 shows that the definition of T being a Hilbert-Schmidt operator, and the value of $\|T\|_2$, do not depend on the choice of orthonormal basis for H . We set

$$\mathcal{HS}(H) := \{T \in \mathcal{B}(H) : T \text{ is a Hilbert-Schmidt operator}\}.$$

4.2. Compact operators on Hilbert spaces

If $H \neq \{0\}$ is finite-dimensional, we set $\mathcal{HS}(H) := \mathcal{B}(H)$ (so we also have $\mathcal{HS}(H) = \mathcal{K}(H) = \mathcal{F}(H)$), and for $T \in \mathcal{B}(H)$, we set

$$\|T\|_2 := \left(\sum_{j=1}^n \|T(u_j)\|^2 \right)^{1/2}$$

for any choice of o.n.b. $\{u_1, \dots, u_n\}$ for H .

Proposition 4.2.8. *Let $H \neq \{0\}$ be a separable Hilbert space (over $\mathbb{K}$). Then $\mathcal{HS}(H)$ is a subspace of $\mathcal{K}(H)$, which contains $\mathcal{F}(H)$ and is closed under the adjoint operation.*

Moreover, the map $T \mapsto \|T\|_2$ is a norm on $\mathcal{HS}(H)$, which satisfies

$$\|T\| \leq \|T\|_2$$

for every $T \in \mathcal{HS}(H)$.

Proof. We assume that H is infinite-dimensional, the finite-dimensional case being easier and left to the reader.

We first note that it is evident from the proof of Lemma 4.2.5 that $T^* \in \mathcal{HS}(H)$ whenever $T \in \mathcal{HS}(H)$.

Let $\mathcal{B} = \{u_j\}_{j \in \mathbb{N}}$ be an orthonormal basis for H , and let $T, T' \in \mathcal{HS}(H)$. Define $\xi, \xi' \in \ell^2(\mathbb{N})$ by

$$\xi(j) := \|T(u_j)\| \quad \text{and} \quad \xi'(j) := \|T'(u_j)\| \quad \text{for each } j \in \mathbb{N},$$

so that $\|\xi\|_2 = \|T\|_2$ and $\|\xi'\|_2 = \|T'\|_2$. Using the triangle inequality, first in H , and then in $\ell^2(\mathbb{N})$, we get

$$\begin{aligned} \sum_{j=1}^{\infty} \|(T + T')(u_j)\|^2 &\leq \sum_{j=1}^{\infty} \left(\|T(u_j)\| + \|T'(u_j)\| \right)^2 = \|\xi + \xi'\|_2^2 \\ &\leq (\|\xi\|_2 + \|\xi'\|_2)^2 = (\|T\|_2 + \|T'\|_2)^2 < \infty. \end{aligned}$$

This shows that $T + T' \in \mathcal{HS}(H)$ and

$$\|T + T'\|_2 \leq \|T\|_2 + \|T'\|_2.$$

Moreover, one easily checks that $\lambda T \in \mathcal{HS}(H)$ and $\|\lambda T\|_2 = |\lambda| \|T\|_2$ for every $\lambda \in \mathbb{K}$. If $\|T\|_2 = 0$, then we get that $\|T(u_j)\| = 0$ for every $j \in \mathbb{N}$, and this clearly implies that $T = 0$.

Hence, we have shown so far that $\mathcal{HS}(H)$ is a subspace of $\mathcal{B}(H)$ which is closed under the adjoint operation, and that $\|\cdot\|_2$ is a norm on $\mathcal{HS}(H)$.

4. On compact operators

To show that $\|T\| \leq \|T\|_2$, let $x \in H \setminus \{0\}$. Set $v_1 = \frac{1}{\|x\|}x$ and let $\{v_j\}_{j \geq 2}$ be an orthonormal basis for $\{x\}^\perp$. Then $\{v_j\}_{j \in \mathbb{N}}$ is an orthonormal basis for H , so we get

$$\|T(x)\|^2 = \|x\|^2 \|T(v_1)\|^2 \leq \|x\|^2 \sum_{j=1}^{\infty} \|T(v_j)\|^2 = \|T\|_2^2 \|x\|^2.$$

Thus, $\|T\| \leq \|T\|_2$.

Next, we show that $T \in \mathcal{K}(H)$. For each $n \in \mathbb{N}$, let P_n denote the orthogonal projection of H on $\text{Span}(\{u_1, \dots, u_n\})$ and set $T_n := TP_n$. Then we have

$$\sum_{j=1}^{\infty} \|T_n(u_j)\|^2 = \sum_{j=1}^n \|T(u_j)\|^2 < \infty,$$

so $T_n \in \mathcal{HS}(H)$ for each $n \in \mathbb{N}$. Hence,

$$\|T - T_n\| \leq \|T - T_n\|_2 = \left(\sum_{j=n+1}^{\infty} \|T(u_j)\|^2 \right)^{1/2} \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Since $T_n \in \mathcal{F}(H)$ for each n , Theorem 4.1.7 gives that $T \in \mathcal{K}(H)$. Hence, $\mathcal{HS}(H) \subseteq \mathcal{K}(H)$.

It only remains to show that $\mathcal{F}(H) \subseteq \mathcal{HS}(H)$, but we leave this as an easy exercise. ■

Remark 4.2.9. For additional properties of $\mathcal{HS}(H)$, see Exercise 4.6.

Remark 4.2.10. If $H \neq \{0\}$ is finite-dimensional, $\mathcal{B}$ is an orthonormal basis for H , then letting $A = [a_{i,j}]$ denotes the matrix of T w.r.t. $\mathcal{B}$, one readily checks that

$$\|T\|_2 = \left(\sum_{i,j=1}^n |a_{i,j}|^2 \right)^{1/2},$$

i.e., $\|T\|_2$ coincides with the so-called Fröbenius-norm of A . □

Example 4.2.11. Set $H = L^2([a, b], \mathcal{A}, \mu)$, where $\mathcal{A}$ denotes the Lebesgue measurable subsets of a closed interval $[a, b]$ and μ is the Lebesgue measure on $\mathcal{A}$. Let $K : [a, b] \times [a, b] \rightarrow \mathbb{C}$ be a continuous function and let $T_K \in \mathcal{B}(H)$ denote the associated integral operator on H , which is the extension of the integral operator $T_K : C([a, b]) \rightarrow C([a, b])$ given by

$$[T_K(f)](s) = \int_a^b K(s, t) f(t) \, dt \quad \text{for } f \in C([a, b]) \text{ and } s \in [a, b].$$

cf. Example 1.3.6. Then we claim that T_K is a Hilbert-Schmidt operator on H . It will then follow that T_K is compact by Proposition 4.2.8.

4.2. Compact operators on Hilbert spaces

To prove this claim, we start by picking an orthonormal basis $\mathcal{B} = \{[u_j]\}_{j \in \mathbb{N}}$ for H , where each u_j is a continuous function on $[a, b]$. (One may for example construct $\mathcal{B}$ by applying the Gram-Schmidt orthonormalization process to the monomials $\{t^{j-1} : j \in \mathbb{N}\}$). We note that $\overline{\mathcal{B}} := \{[\overline{u_j}]\}_{j \in \mathbb{N}}$ is then also an orthonormal basis for H .

Let now $s \in [a, b]$ and let $k_s \in C([a, b])$ be given by $k_s(t) = K(s, t)$ for all $t \in [a, b]$. Note that for each $j \in \mathbb{N}$, we have

$$[T_K(u_j)](s) = \int_a^b K(s, t) u_j(t) \, dt = \int_{[a, b]} k_s(t) \overline{u_j(t)} \, d\mu(t) = \langle k_s, \overline{u_j} \rangle.$$

Moreover, Parseval's identity gives that

$$\|k_s\|_2 = \left(\sum_{j=1}^{\infty} |\langle k_s, \overline{u_j} \rangle|^2 \right)^{1/2}.$$

Thus, we obtain that

$$\sum_{j=1}^{\infty} |[T_K(u_j)](s)|^2 = \sum_{j=1}^{\infty} |\langle k_s, \overline{u_j} \rangle|^2 = \|k_s\|_2^2.$$

Now, using this and the Monotone Convergence Theorem, we get

$$\begin{aligned} \sum_{j=1}^{\infty} \|T_K(u_j)\|_2^2 &= \sum_{j=1}^{\infty} \int_{[a, b]} |[T_K(u_j)](s)|^2 \, d\mu(s) \\ &= \int_{[a, b]} \left(\sum_{j=1}^{\infty} |[T_K(u_j)](s)|^2 \right) \, d\mu(s) \\ &= \int_{[a, b]} \|k_s\|_2^2 \, d\mu(s) \\ &= \int_{[a, b]} \left(\int_{[a, b]} |k_s(t)|^2 \, d\mu(t) \right) \, d\mu(s) \\ &= \int_a^b \int_a^b |K(s, t)|^2 \, dt \, ds < \infty, \end{aligned}$$

which shows that $T_K \in \mathcal{HS}(H)$ with $\|T_K\|_2 \leq (\int_a^b \int_a^b |K(s, t)|^2 \, ds \, dt)^{1/2}$.

In the previous example, one may allow the kernel K to be discontinuous and still obtain an Hilbert-Schmidt operator T_K , as long as K is square-integrable w.r.t. to the product Lebesgue measure on $[a, b] \times [a, b]$. We leave this as an exercise.

4.3 The spectral theorem for a compact self-adjoint operator

Throughout this section we let H denote a Hilbert space (over $\mathbb{K}$) different from $\{0\}$. Our main goal is to generalize the spectral theorem for symmetric real matrices known from linear algebra, and prove that every compact self-adjoint compact operator T on H is diagonalizable in the sense that there exists an orthonormal basis for H consisting of eigenvectors of T .

We begin with a series of lemmas.

Lemma 4.3.1. *Assume $T \in \mathcal{K}(H)$ has a nonzero eigenvalue $\lambda \in \mathbb{K}$. Then the associated eigenspace $E_\lambda := \ker(T - \lambda I)$ is finite-dimensional.*¹

Proof. Assume for contradiction that E_λ is infinite-dimensional. We may then find an orthonormal sequence $\{v_n\}_{n \in \mathbb{N}}$ in E_λ . By compactness of T , $\{T(v_n)\}_{n \in \mathbb{N}}$ has a convergent subsequence. So we may as well assume that $\{T(v_n)\}_{n \in \mathbb{N}}$ is convergent, hence that it is a Cauchy sequence. However, we have that

$$\|T(v_n) - T(v_m)\|^2 = \|\lambda v_n - \lambda v_m\|^2 = 2|\lambda|^2 > 0$$

for all $m, n \in \mathbb{N}$. So $\{T(v_n)\}_{n \in \mathbb{N}}$ is not a Cauchy sequence, giving a contradiction. ■

Lemma 4.3.2. *Let $T \in \mathcal{B}(H)$ be self-adjoint, and assume T has an eigenvalue $\lambda \in \mathbb{K}$. Then $\lambda \in \mathbb{R}$.*

Moreover, if λ' is an eigenvalue of T distinct from λ , then $E_\lambda \perp E_{\lambda'}$, i.e., $\langle x, y \rangle = 0$ whenever $x \in E_\lambda$ and $y \in E_{\lambda'}$.

Proof. Let $x \in E_\lambda$. If $\|x\| = 1$, then we have

$$\lambda = \lambda \langle x, x \rangle = \langle \lambda x, x \rangle = \langle T(x), x \rangle \in W_T \subseteq \mathbb{R},$$

so $\lambda \in \mathbb{R}$. Moreover, assume that λ' is an eigenvalue of T distinct from λ , and let $y \in E_{\lambda'}$. Then we have that $\lambda' \in \mathbb{R}$, so

$$\lambda \langle x, y \rangle = \langle T(x), y \rangle = \langle x, T(y) \rangle = \lambda' \langle x, y \rangle.$$

Since $\lambda' \neq \lambda$, we get that $\langle x, y \rangle = 0$. ■

The next lemma is crucial.

¹If there is a danger of confusion, we will write E_λ^T instead of E_λ .

4.3. The spectral theorem for a compact self-adjoint operator

Lemma 4.3.3. *Let $T \in \mathcal{K}(H)$ be self-adjoint. Then T has an eigenvalue $\lambda \in \mathbb{R}$ such that $|\lambda| = \|T\|$.*

Proof. If $T = 0$, then the assertion is trivial. So assume that $T \neq 0$. Using Theorem 2.5.9, we can find a sequence $\{x_n\}_{n \in \mathbb{N}}$ of unit vectors in H such that $|\langle T(x_n), x_n \rangle| \rightarrow \|T\|$ as $n \rightarrow \infty$. Since $\langle T(x_n), x_n \rangle \in \mathbb{R}$ for every n , we can assume (by passing to a subsequence and relabelling) that

$$\langle T(x_n), x_n \rangle \rightarrow \lambda \text{ as } n \rightarrow \infty, \text{ where } \lambda = \pm\|T\|. \quad (4.3.1)$$

Moreover, since T is compact, we can also assume (by passing again to a subsequence and relabelling) that $T(x_n) \rightarrow y$ as $n \rightarrow \infty$ for some $y \in H$. Note that the Cauchy-Schwarz inequality gives that

$$|\langle T(x_n), x_n \rangle| \leq \|T(x_n)\| \quad \text{for every } n \in \mathbb{N},$$

so, letting $n \rightarrow \infty$, we get that $\|y\| \geq |\lambda| > 0$, so $y \neq 0$.

Now, using that T is self-adjoint, λ is real, $\|x_n\| = 1$, and (4.3.1), we get

$$\begin{aligned} \|T(x_n) - \lambda x_n\|^2 &= \langle T(x_n) - \lambda x_n, T(x_n) - \lambda x_n \rangle \\ &= \|T(x_n)\|^2 - 2\lambda \langle T(x_n), x_n \rangle + \lambda^2 \|x_n\|^2 \\ &\leq \|T\|^2 - 2\lambda \langle T(x_n), x_n \rangle + \lambda^2 \\ &= 2\lambda (\lambda - \langle T(x_n), x_n \rangle) \\ &\rightarrow 0 \text{ as } n \rightarrow \infty. \end{aligned}$$

Thus, $\|T(x_n) - \lambda x_n\| \rightarrow 0$ as $n \rightarrow \infty$, and this gives that

$$\|y - \lambda x_n\| \leq \|y - T(x_n)\| + \|T(x_n) - \lambda x_n\| \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Hence,

$$T(y) = \lim_{n \rightarrow \infty} T(\lambda x_n) = \lambda \lim_{n \rightarrow \infty} T(x_n) = \lambda y.$$

Since $y \neq 0$, λ is an eigenvalue of T , as we wanted to show. ■

We are now ready to state and prove the *spectral theorem for a compact self-adjoint operator* T . Intuitively, we could hope to be able to construct an orthonormal basis of eigenvectors for T by using Lemma 4.3.3 repeatedly as follows:

Start by picking a unit eigenvector v_0 of T associated to the eigenvalue $\lambda_0 = \pm\|T\|$. Next, consider the restriction T_1 of T to the closed subspace $M_1 := \{v_0\}^\perp$. Pick a unit eigenvector $v_1 \in M_1$ of T_1 associated to the eigenvalue $\lambda_1 = \pm\|T_1\|$. Then continue this process inductively.

There are several technicalities involved in working out the details of such an approach. We will follow a more pedestrian route, which will also provide us with more information about T .

4. On compact operators

Theorem 4.3.4. *Let $T \in \mathcal{K}(H)$ be self-adjoint, $T \neq 0$. Then there exists an orthonormal basis $\mathcal{E}$ for H which consists of eigenvectors of T .*

More precisely, the following facts hold:

- (a) *The set L consisting of all nonzero eigenvalues of T is a nonempty, countable subset of the interval $[-\|T\|, \|T\|]$, containing $\|T\|$ or $-\|T\|$.*
- (b) *If L is countably infinite, and $\{\lambda_k\}_{k \in \mathbb{N}}$ is an enumeration of L , then we have $\lim_{k \rightarrow \infty} \lambda_k = 0$.*
- (c) *The eigenspace $E_\lambda := \ker(T - \lambda I)$ is finite-dimensional for each $\lambda \in L$. (The dimension of E_λ is called the (geometric) multiplicity of λ .)*
- (d) *For each $\lambda \in L$, let $\mathcal{E}_\lambda$ be an orthonormal basis for E_λ , and set*

$$\mathcal{E}' := \bigcup_{\lambda \in L} \mathcal{E}_\lambda.$$

Then $\mathcal{E}'$ is an orthonormal basis for $\overline{T(H)} = \ker(T)^\perp$, which is countable.

- (e) *If $\ker(T) = \{0\}$, set $\mathcal{E}_0 := \emptyset$; otherwise, let $\mathcal{E}_0$ be an orthonormal basis for $\ker(T)$. Then $\mathcal{E} := \mathcal{E}_0 \cup \mathcal{E}'$ is an orthonormal basis for H which consists of eigenvectors of T .*
- (f) *Let P_λ denote the orthogonal projection of H on E_λ for each $\lambda \in L$. Then $P_\lambda P_{\lambda'} = 0$ whenever $\lambda \neq \lambda'$ belong to L . Moreover, T has a spectral decomposition*

$$T = \sum_{\lambda \in L} \lambda P_\lambda \quad (\text{w.r.t. operator norm}), \text{ meaning that}$$

$\lim_{n \rightarrow \infty} \|T - \sum_{k=1}^n \lambda_k P_{\lambda_k}\| = 0$ if L is countably infinite and $\{\lambda_k\}_{k \in \mathbb{N}}$ is an enumeration of L , as in (b).

Proof. (a): The set L is a subset of $\mathbb{R}$ by Lemma 4.3.2, which contains $\|T\|$ or $-\|T\|$ by Lemma 4.3.3. If $\lambda \in L$, and v is an associated eigenvector in H_1 , we have

$$|\lambda| = |\langle \lambda v, v \rangle| = |\langle T(v), v \rangle| \leq \|T\|.$$

Thus, $L \subseteq [-\|T\|, \|T\|]$

To show that L is countable, let $\varepsilon > 0$ and consider the subset of L given by $L_\varepsilon := \{\lambda \in L : |\lambda| \geq \varepsilon\}$. Then L_ε is finite.

4.3. The spectral theorem for a compact self-adjoint operator

Indeed, assume L_ε is nonempty. Then for each $\lambda \in L$, we can pick $v_\lambda \in H_1$ such that $T(v_\lambda) = \lambda v_\lambda$; for $\lambda, \lambda' \in L_\varepsilon, \lambda \neq \lambda'$, we then have $\lambda v_\lambda \perp \lambda' v_{\lambda'}$ by Lemma 4.3.2, so we get

$$\|T(v_\lambda) - T(v_{\lambda'})\|^2 = \|\lambda v_\lambda - \lambda' v_{\lambda'}\|^2 = |\lambda|^2 + |\lambda'|^2 \geq 2\varepsilon^2.$$

Hence, if L_ε was infinite, we could find a sequence in H_1 which T maps into a sequence with no convergent subsequence, contradicting the compactness of T . Thus, L_ε is finite.

Now, since $L = \bigcup_{n \in \mathbb{N}} L_{1/n}$, it follows that L is countable.

(b): Assume L is countably infinite and $\{\lambda_k\}_{k \in \mathbb{N}}$ is an enumeration of L . Let $\varepsilon > 0$ be given. Then, as in (a), we get that the set $K := \{k \in \mathbb{N} : |\lambda_k| \geq \varepsilon\}$ is finite. So there exists $N \in \mathbb{N}$ such that $K \subseteq \{1, \dots, N\}$. For every $k \geq N + 1$, we then have $|\lambda_k| < \varepsilon$. This shows that $\lim_{k \rightarrow \infty} \lambda_k = 0$.

(c): This is a consequence of Lemma 4.3.1.

(d): We first remark that since T is self-adjoint, we have

$$\overline{T(H)} = \overline{T^*(H)} = (\ker T)^\perp.$$

Next, it follows from Lemma 4.3.2 that $\mathcal{E}_\lambda \perp \mathcal{E}_{\lambda'}$ whenever $\lambda \neq \lambda'$ belong to L . So it is clear that $\mathcal{E}'$ is an orthonormal set in H , which is countable since each $\mathcal{E}_\lambda$ is finite and L is countable. Hence, $\mathcal{E}'$ is a countable orthonormal basis for $M := \overline{\text{Span}(\mathcal{E})}$, and it remains only to show that $M = \ker(T)^\perp$, i.e., that $M^\perp = \ker(T)$.

- $\ker(T) \subseteq \overline{M^\perp}$: Assume $y \in \ker(T)$. Then for each $\lambda \in L$ and $v \in \mathcal{E}_\lambda$, we have

$$\lambda \langle v, y \rangle = \langle T(v), y \rangle = \langle v, T(y) \rangle = \langle v, 0 \rangle = 0.$$

Since $\lambda \neq 0$, this shows that $y \in (\mathcal{E}')^\perp = M^\perp$.

- $\overline{M^\perp} \subseteq \ker(T)$: It is easy to check that M is invariant under T . Hence, $M^\perp$ is invariant under $T^* = T$ (cf. Exercise 2.21). We may therefore consider the restriction S of T to $M^\perp$. Then $S \in \mathcal{K}(M^\perp)$: if not, then there would exist a bounded sequence in $M^\perp$, hence in H , which is mapped by S , hence by T , to a sequence with no convergent subsequence, contradicting the compactness of T . Moreover, S is self-adjoint (this is an easy exercise).

Now, assume that $S \neq 0$. Then Lemma 4.3.3 gives that S has a nonzero eigenvalue μ . This implies that μ is a nonzero eigenvalue of T , i.e., $\mu \in L$. But if $v \in M^\perp$ is an eigenvector for S associated with μ ,

4. On compact operators

we then have that $v \in E_\mu \subseteq M$, so $v \in M \cap M^\perp = \{0\}$, contradicting that $v \neq 0$ (since v is an eigenvector).

This means that S has to be 0. Thus we get $T(y) = S(y) = 0$ for all $y \in M^\perp$, as desired.

(e): If $\ker(T) = \{0\}$, then we get from (d) that $\mathcal{E} = \mathcal{E}'$ is an orthonormal basis for $\ker(T)^\perp = \{0\}^\perp = H$. If $\ker(T) \neq \{0\}$, then we have $\mathcal{E}_0 \subseteq \ker(T)$ and $\mathcal{E}' \subseteq \ker(T)^\perp$, so it is clear that $\mathcal{E}$ is an orthonormal set. Moreover, we have that

$$H = \overline{\text{Span}(\mathcal{E})}.$$

Indeed, let $x \in H$. Then we may write

$$x = x_M + x_{M^\perp},$$

where $x_M \in M = \overline{\text{Span}(\mathcal{E}')}$ and $x_{M^\perp} \in M^\perp = \ker(T) = \overline{\text{Span}(\mathcal{E}_0)}$. So we may choose $\{x_n\}_{n \in \mathbb{N}} \subseteq \text{Span}(\mathcal{E}')$ and $\{y_n\}_{n \in \mathbb{N}} \subseteq \text{Span}(\mathcal{E}_0)$ such that $\lim_{n \rightarrow \infty} x_n = x_M$ and $\lim_{n \rightarrow \infty} y_n = x_{M^\perp}$. This gives that

$$\lim_{n \rightarrow \infty} (x_n + y_n) = x_M + x_{M^\perp} = x.$$

Hence, $x \in \overline{\text{Span}(\mathcal{E})}$. This shows that $\mathcal{E}$ is an orthonormal basis for H .

(f): The first assertion follows readily from the fact that $E_\lambda \perp E_{\lambda'}$ whenever $\lambda \neq \lambda'$, cf. Lemma 4.3.2. Next, we consider the case where L is countably infinite and $\{\lambda_k\}_{k \in \mathbb{N}}$ is an enumeration of L , leaving the easier case where L is finite to the reader.

For each $k \in \mathbb{N}$, set $n_k := \dim(E_{\lambda_k}) < \infty$, and let $\{v_{k,1}, \dots, v_{k,n_k}\}$ be an enumeration of $\mathcal{E}_{\lambda_k}$. Then we have

$$\mathcal{E}' = \bigcup_{k \in \mathbb{N}} \mathcal{E}_{\lambda_k} = \{v_{k,l} : k \in \mathbb{N}, 1 \leq l \leq n_k\}.$$

Consider $x \in H$. Since $T(x) \in \overline{T(H)}$ and $\mathcal{E}'$ is an orthonormal basis for $\overline{T(H)}$, we get from Corollary 2.3.11 that

$$\begin{aligned} T(x) &= \lim_{m \rightarrow \infty} \sum_{k=1}^m \sum_{l=1}^{n_k} \langle T(x), v_{k,l} \rangle v_{k,l} = \lim_{m \rightarrow \infty} \sum_{k=1}^m \sum_{l=1}^{n_k} \langle x, T(v_{k,l}) \rangle v_{k,l} \\ &= \lim_{m \rightarrow \infty} \sum_{k=1}^m \lambda_k \left(\sum_{l=1}^{n_k} \langle x, v_{k,l} \rangle v_{k,l} \right) = \sum_{k=1}^{\infty} \lambda_k P_{\lambda_k}(x). \end{aligned}$$

Let now $\varepsilon > 0$. We have to show that there exists $N \in \mathbb{N}$ such that $\|T - \sum_{k=1}^n \lambda_k P_{\lambda_k}\| \leq \varepsilon$ for all $n \geq N$.

4.3. The spectral theorem for a compact self-adjoint operator

Using (b), we can choose $N \in \mathbb{N}$ such that $|\lambda_k| < \varepsilon$ for all $k > N$. Then for all $n \geq N$ and all $x \in H$, using continuity of the norm in H and Pythagoras' identity, we get

$$\begin{aligned} \left\| \left(T - \sum_{k=1}^n \lambda_k P_{\lambda_k} \right)(x) \right\|^2 &= \left\| \sum_{k=n+1}^{\infty} \lambda_k P_{\lambda_k}(x) \right\|^2 = \sum_{k=n+1}^{\infty} |\lambda_k|^2 \|P_{\lambda_k}(x)\|^2 \\ &\leq \varepsilon^2 \sum_{k=n+1}^{\infty} \|P_{\lambda_k}(x)\|^2 \leq \varepsilon^2 \|x\|^2 \end{aligned}$$

and the assertion follows. ■

Sometimes, the following consequence of the spectral theorem is useful.

Corollary 4.3.5. *Let $T \in \mathcal{K}(H)$ be self-adjoint, $T \neq 0$. Then there exist an orthonormal family $\{u_j\}_{j \in J}$ in H and a family $\{\rho_j\}_{j \in J}$ in $\mathbb{R} \setminus \{0\}$, both indexed over the same countable set J , such that*

$$T(x) = \sum_{j \in J} \rho_j \langle x, u_j \rangle u_j \quad \text{for all } x \in H. \quad (4.3.2)$$

Each ρ_j is an eigenvalue of T and u_j is an associated eigenvector.

Proof. We use the notation from Theorem 4.3.4). Since $\mathcal{E}' = \cup_{\lambda \in L} \mathcal{E}_\lambda$ is countable, we may let $\{u_j : j \in J\}$ be an enumeration of $\mathcal{E}'$, where the index set J is countable. Moreover, for each $j \in J$, we may then let $\rho_j \in \mathbb{R} \setminus \{0\}$ denote the eigenvalue of T corresponding to u_j . (The family $\{\rho_j\}_{j \in J}$ gives a list of all nonzero eigenvalues of T , repeated according to their multiplicities.) Since $\mathcal{E}' = \{u_j : j \in J\}$ is an orthonormal basis for $\overline{T(H)}$, as in the proof of (f), we get

$$T(x) = \sum_{j \in J} \langle T(x), u_j \rangle u_j = \sum_{j \in J} \langle x, T(u_j) \rangle u_j = \sum_{j \in J} \rho_j \langle x, u_j \rangle u_j$$

for all $x \in H$. ■

Let us say that an operator $T \in \mathcal{B}(H)$ is *diagonalizable* if there exists an orthonormal basis for H whose elements are eigenvectors for T . Then the spectral theorem (Theorem 4.3.4) tells us that T is diagonalizable whenever T is compact and self-adjoint. Let us now say that an operator $T \in \mathcal{B}(H)$ is *normal* if T^* commutes with T , i.e., we have $T^*T = TT^*$. Clearly, self-adjoint operators and unitary operators are normal (but note that unitary operators, being invertible, are never compact when H is infinite-dimensional. Compact operators on H that are diagonalizable can be described as follows.

4. On compact operators

Theorem 4.3.6. *Assume T is a compact operator on H . If $\mathbb{K} = \mathbb{R}$, then T is diagonalizable if and only if T is self-adjoint. On the other hand, if $\mathbb{K} = \mathbb{C}$, then T is diagonalizable if and only if T is normal.*

This theorem can be deduced from Theorem 4.3.4. We leave the proof to the reader (cf. Exercises 4.12 and 4.13).

As another corollary of Theorem 4.3.4, we end this section by showing how an analogue of the singular value decomposition (SVD) for matrices may be obtained for compact operators.

Let $S \in \mathcal{K}(H)$, $S \neq 0$. Then $T := S^*S$ is self-adjoint and compact, and $T \neq 0$ (as $\|T\| = \|S^*S\| = \|S\|^2 \neq 0$). Hence, the spectral theorem (Theorem 4.3.4) gives that we can find a countable orthonormal basis $\{v_j\}_{j \in N}$ for

$$\overline{T(H)} = \ker(T)^\perp = \ker(S^*S)^\perp = \ker(S)^\perp$$

consisting of eigenvectors for T . For each $j \in N$, let μ_j denote the eigenvalue of T associated with v_j . Note that

$$\mu_j = \langle \mu_j v_j, v_j \rangle = \langle T(v_j), v_j \rangle = \langle S(v_j), S(v_j) \rangle = \|S(v_j)\|^2 \geq 0$$

for every $j \in N$. Since each μ_j is nonzero, we get that all μ_j 's are positive. For each $j \in N$, set

$$\sigma_j := \sqrt{\mu_j} > 0 \quad \text{and} \quad u_j := \frac{1}{\sigma_j} S(v_j).$$

The σ_j 's are called the *singular values* of S . For all $j, k \in N$ we have

$$\begin{aligned} \langle u_j, u_k \rangle &= \frac{1}{\sigma_j \sigma_k} \langle S(v_j), S(v_k) \rangle = \frac{1}{\sigma_j \sigma_k} \langle T(v_j), v_k \rangle \\ &= \frac{\mu_j}{\sigma_j \sigma_k} \langle v_j, v_k \rangle = \begin{cases} 1 & \text{if } j = k, \\ 0 & \text{otherwise,} \end{cases} \end{aligned}$$

so $\{u_j : j \in N\}$ is an orthonormal set in the range of S . Further, we have the following decomposition of S :

$$S(x) = \sum_{j \in N} \sigma_j \langle x, v_j \rangle u_j \quad \text{for all } x \in H. \quad (4.3.3)$$

Indeed, let $x \in H$ and set $M := \overline{T(H)}$, so $M^\perp = \ker(S)$. With $z := x - P_M(x) \in M^\perp$, we get that

$$x = P_M(x) + z = \sum_{j \in N} \langle x, v_j \rangle v_j + z,$$

so

$$S(x) = \sum_{j \in N} \langle x, v_j \rangle S(v_j) + S(z) = \sum_{j \in N} \sigma_j \langle x, v_j \rangle u_j,$$

as asserted in (4.3.3).

It readily follows that $\{u_j : j \in N\}$ is an orthonormal basis for $\overline{S(H)}$.

Finally we remark that the spectral theorem also gives that $\sigma_j = \sqrt{\mu_j} \rightarrow 0$ as $j \rightarrow \infty$ when N is countably infinite, and that

$$\|S\| = \|S^*S\|^{1/2} = \|T\|^{1/2} = \max\{\mu_j : j \in N\}^{1/2} = \max\{\sigma_j : j \in N\}.$$

4.4 The Fredholm Alternative

A useful application of linear algebra, and one of its original motivation, is the study of systems of linear equations, i.e., of equations of the type $Ax = b$, where $A \in M_{m \times n}(\mathbb{K})$, $b \in \mathbb{K}^m$ and the (unknown) vector x belongs to $\mathbb{K}^n$. More generally, one may consider equations of the form

$$T(v) = w \tag{4.4.1}$$

where V, W are vector spaces (over $\mathbb{K}$), $T \in \mathcal{L}(V, W)$, $w \in W$ and the (unknown) vector x belongs to V . Whether such an equation is consistent, i.e., has some solution(s), relies on whether w lies in the range of T . If this is the case, and $v_0 \in V$ is any vector satisfying (4.4.1), i.e., such that $T(v_0) = w$, then it follows readily that the solution set of (4.4.1) is given by

$$v_0 + \ker(T) := \{v_0 + u \mid u \in \ker(T)\} \tag{4.4.2}$$

where $v_0 \in V$ is any vector satisfying (4.4.1), i.e., such that $T(v_0) = w$.

In the rest of this section, we consider the case where $V = W = H$ is a Hilbert space ($\neq \{0\}$), and $T \in \mathcal{B}(H)$. We can then exploit the relationship between the fundamental subspaces of T and T^* , e.g.,

$$\overline{T(H)} = \ker(T^*)^\perp \quad \text{and} \quad \overline{T^*(H)} = \ker(T)^\perp, \tag{4.4.3}$$

cf. Proposition 2.4.10, and study in parallel the equation $T^*(v') = w'$.

Example 4.4.1. Assume that T has closed range (i.e., $T(H)$ is closed). Then we get from (4.4.3) that the equation (4.4.1) is consistent if and only if w is orthogonal to $\ker(T^*)$.

In particular, if T has closed range and $\ker(T^*) = \{0\}$ (i.e., T^* is one-to-one), then T must be surjective, hence (4.4.1) is consistent for all $w \in H$.

Similarly, if T^* has closed range and $\ker(T) = \{0\}$, then T^* is surjective, i.e., the equation $T^*(v') = w'$ is consistent for all $w' \in H$.

4. On compact operators

□

Example 4.4.2. Assume that T is surjective. Then we get from (4.4.3) that $\ker(T^*) = \{0\}$, hence that the equation $T^*(v') = w'$ has either no solution or a unique solution.

Similarly, if T^* is surjective, then $\ker(T) = \{0\}$, and (4.4.1) has either no solution or a unique solution.

□

A problem is that bounded operators in general do not have a closed range (see for example Exercises 4.16 and 4.17). Moreover, it may often be a difficult task to decide whether the range of some given $T \in \mathcal{B}(H)$ is closed or not. However, we note that if $T \in \mathcal{B}(H)$ has finite-rank, then it has closed range (since $T(H)$ is finite-dimensional). In the case where H is finite-dimensional, much more can be said.

The following terminology will be useful.

Definition 4.4.3. Let $F \in \mathcal{B}(H)$. We will say that F satisfies the *Fredholm alternative* if one of the following two (mutually exclusive) situations occurs:

- (a) $\ker(F) = \ker(F^*) = \{0\}$, and the equations $F(v) = w$, $F^*(v') = w'$ have both a unique solution for all $w, w' \in H$. [This means that both F and F^* are bijective.]
- (b) $1 \leq \dim(\ker(F^*)) = \dim(\ker(F)) < \infty$, the equation $F(v) = w$ is consistent if and only if $w \in \ker(F^*)^\perp$, and the equation $F^*(v') = w'$ is consistent if and only if $w' \in \ker(F)^\perp$.

Operators satisfying the Fredholm alternative may be characterized as follows.

Proposition 4.4.4. Let $F \in \mathcal{B}(H)$. Then F satisfies the Fredholm alternative if and only if the following conditions hold:

- F has closed range;
- F^* has closed range;
- $\dim(\ker(F^*)) = \dim(\ker(F)) < \infty$.

Proof. Assume that the above conditions hold. Then the first two conditions give that

$$F(H) = \ker(F^*)^\perp \text{ and } F^*(H) = \ker(F)^\perp. \quad (4.4.4)$$

4.4. The Fredholm Alternative

Now, in view of the third condition, there are only two possibilities: either $\ker(F^*) = \ker(F) = \{0\}$ or $1 \leq \dim(\ker(F^*)) = \dim(\ker(F)) < \infty$. In the first case, we readily get that both F and F^* are bijective, hence that alternative (a) holds. In the second case, it is immediate from (4.4.4) that alternative (b) holds. Thus, F satisfies the Fredholm alternative.

Conversely, assume F satisfies the Fredholm alternative. Then the third condition clearly holds. Moreover, under alternative (a), we have $F(H) = F^*(H) = H$, while, under alternative (b) we have $F(H) = \ker(T^*)^\perp$ and $F^*(H) = \ker(T)^\perp$. Hence, both F and F^* have closed ranges. ■

Example 4.4.5. Assume that H is finite-dimensional and $F \in \mathcal{B}(H)$, i.e., $F : H \rightarrow H$ is linear. Then F satisfies the Fredholm alternative.

Since $F(H)$ and $F^*(H)$ are finite-dimensional, both F and F^* have closed range. Thus Proposition 4.4.4 will give us that F satisfies the Fredholm alternative if we can show that $\dim(\ker(F^*)) = \dim(\ker(F))$. But this follows from the dimension formula for F , i.e.,

$$\dim(\ker(F)) + \dim(F(H)) = \dim(H)$$

and the well-known fact² that for any subspace M of H , we have

$$\dim(M) + \dim(M^\perp) = \dim(H).$$

Indeed, we get

$$\dim(\ker(F^*)) = \dim(F(H)^\perp) = \dim(H) - \dim(F(H)) = \dim(\ker(F)),$$

as wanted. □

Let T_K be an integral operator on $L^2([a, b])$, where $K : [a, b] \times [a, b] \rightarrow \mathbb{C}$ is a continuous function, and $\mu \in \mathbb{C}$. An equation of the form $T_K(f) = g$, where $f, g \in L^2([a, b])$, is called a *Fredholm integral equation of the first kind*. Moreover, an equation of the form

$$(T_K - \mu I)(f) = g, \text{ i.e., } T_K(f) - \mu f = g,$$

where $f, g \in L^2([a, b])$ and $\mu \in \mathbb{C} \setminus \{0\}$, is called a *Fredholm integral equation of the second kind*. Such equations were studied by I. Fredholm at the beginning of the 20th century. They arise in some practical problems in signal theory and in physics.

We will be here interested in equations of the second kind. Fredholm showed that if $\mu \neq 0$, then the operator $T_K - \mu I$ satisfies the Fredholm

²This fact follows by applying the dimension formula to the projection map P_M .

4. On compact operators

alternative. Now, as we have seen in Example 4.2.11, T_K is a compact operator. It can in fact be shown that if T is a compact operator on H and $\mu \in \mathbb{K} \setminus \{0\}$, then any operator of the form $T - \mu I$ satisfies the Fredholm alternative.

We will first illustrate how the spectral theorem can be applied to give a direct proof of this fact when T is a compact self-adjoint operator.

Theorem 4.4.6. *Assume $T \in \mathcal{K}(H)$ is self-adjoint and $\mu \in \mathbb{K} \setminus \{0\}$. Then $F = T - \mu I$ satisfies the Fredholm alternative.*

Proof. We divide the proof in two parts.

1) Assume first that μ is an eigenvalue of T , i.e., $\ker(T - \mu I) \neq \{0\}$. Then $\mu \in \mathbb{R}$, so $F^* = F$. Moreover, as $\mu \neq 0$, we have that $T \neq 0$, and the spectral theorem tells us that $1 \leq \dim(\ker(F)) = \dim(\ker(T - \mu I)) < \infty$. Hence, to show that (b) in Definition 4.4.12 holds, it remains only to prove that the equation $F(x) = y$ is consistent if and only if $y \in \ker(F)^\perp$. This means that we have to prove that the equation

$$T(x) - \mu x = y \tag{4.4.5}$$

is consistent if and only if $\langle y, z \rangle = 0$ for all $z \in E_\mu^T := \ker(T - \mu I)$.

To prove this, let $\mathcal{E}' = \{u_j\}_{j \in J}$ be an enumeration of the orthonormal basis for $\overline{T(H)}$ obtained in the spectral theorem for T , and let $\rho_j \in \mathbb{R} \setminus \{0\}$ denote the eigenvalue of T corresponding to each u_j (as in the proof of Corollary 4.3.5).

Since H is the direct sum of $\ker(T)$ and $\ker(T)^\perp = \overline{T(H)}$, we may write $y \in H$ as

$$y = y_0 + \sum_{j \in J} \langle y, u_j \rangle u_j,$$

where y_0 denote the orthogonal projection of y onto $\ker(T)$. Likewise, we may assume that the (unknown) vector x in equation (4.4.5) is written as

$$x = x_0 + \sum_{j \in J} c_j u_j,$$

where $x_0 \in \ker(T)$ and $\{c_j\}_{j \in J} \in \ell^2(J)$ are to be determined, if possible. Plugging this into equation (4.4.5), we get the equivalent equation

$$-\mu x_0 + \sum_{j \in J} (\rho_j - \mu) c_j u_j = y_0 + \sum_{j \in J} \langle y, u_j \rangle u_j.$$

Clearly, we can set $x_0 := (-1/\mu) y_0$, and equation (4.4.5) is then consistent if and only if the sequence $\{c_j\}_{j \in J} \in \ell^2(J)$ can be chosen so that

$$(\rho_j - \mu) c_j = \langle y, u_j \rangle \quad \text{for all } j \in J. \tag{4.4.6}$$

4.4. The Fredholm Alternative

Now, as μ is a nonzero eigenvalue of T , we have that $\mu = \rho_k$ for some $k \in J$. Let $u_{j_1}, \dots, u_{j_n}$ denote the vectors in $\mathcal{E}'$ giving an orthonormal basis for $E_\mu^T = E_{\rho_k}^T$. If $j \notin \{j_1, \dots, j_n\}$, we have $\rho_j \neq \mu$, so

$$c_j := \frac{1}{\rho_j - \mu} \langle y, u_j \rangle$$

will satisfy (4.4.6) for every such j .

On the other hand, if $j \in \{j_1, \dots, j_n\}$, we have $\rho_j - \mu = 0$. Hence, (4.4.6) will be satisfied for $j = j_1, \dots, j_n$ if and only if we have $\langle y, u_j \rangle = 0$ for $j = j_1, \dots, j_n$, i.e., if and only if $\langle y, z \rangle = 0$ for all $z \in E_\mu^T$. Thus, if equation (4.4.5) is consistent, we must have that $\langle y, z \rangle = 0$ for all $z \in E_\mu^T$. Conversely, let's assume that this condition holds. We are then free to choose $c_{j_1}, \dots, c_{j_n}$ as we want. Regardless of this choice, it is not difficult to verify that the constructed sequence $\{c_j\}_{j \in J}$ belongs to $\ell^2(J)$ (cf. Exercise 4.21), so that the vector x associated to this sequence will then give a solution to (4.4.5). Summing up, we have proved that equation (4.4.5) is consistent if and only if $\langle y, z \rangle = 0$ for all $z \in E_\mu^T$. As explained above, this shows that (b) in Definition 4.4.12 holds.

2) Next, assume that μ is not an eigenvalue of T , i.e., $\ker(T - \mu I) = \{0\}$. Then, using the spectral theorem, one can now show that the equation $(T - \mu I)(x) = y$ has a unique solution for all $y \in H$. (You are asked to do this in Exercise 4.20.) Thus, $F = T - \mu I$ is surjective, and this implies that $\ker(F^*) = \ker(T - \bar{\mu} I) = \{0\}$, i.e., $\bar{\mu}$ is not an eigenvalue of T . Arguing as above, we get that the equation $(T - \bar{\mu} I)(x') = y'$, i.e., $(T - \mu I)^*(x') = y'$ has a unique solution for all $y' \in H$. This shows that (a) in Definition 4.4.12 holds in this case. ■

To prove that the Fredholm alternative holds for $T - \mu I$ when $T \in \mathcal{K}(H)$ is not assumed to be self-adjoint and $\mu \in \mathbb{K} \setminus \{0\}$, our plan is to apply Proposition 4.4.4. As a first step, we will show:

Theorem 4.4.7. *Let $T \in \mathcal{K}(H)$ and $\mu \in \mathbb{K} \setminus \{0\}$. Then $F = T - \mu I$ and $F^* = (T - \mu I)^*$ have closed ranges.*

Proof. We have $F^* = (T - \mu I)^* = T^* - \bar{\mu} I$. Since T^* is compact and $\bar{\mu} \neq 0$, it suffices to show that $F = T - \mu I$ has closed range. So let $\{y_n\}_{n \in \mathbb{N}}$ be a sequence in $F(H)$ converging to some $y \in H$. We have to show that $y \in F(H)$.

For each $n \in \mathbb{N}$, we may pick $x_n \in H$ such that $y_n = F(x_n)$. Now, as $H = \ker(F) \oplus \ker(F)^\perp$, we may write

$$x_n = z_n + w_n \quad \text{with } z_n \in \ker(F), w_n \in \ker(F)^\perp.$$

4. On compact operators

We claim that the sequence $\{w_n\}_{n \in \mathbb{N}}$ is bounded:

Assume (for contradiction) that this is not true. Passing to a subsequence if necessary, we may assume that $w_n \neq 0$ for all $n \in \mathbb{N}$ and $\lim_{n \rightarrow \infty} \|w_n\| = \infty$. For each $n \in \mathbb{N}$, set $\tilde{w}_n := \frac{1}{\|w_n\|} w_n$. Then we have

$$\begin{aligned} T(\tilde{w}_n) - \mu \tilde{w}_n &= F(\tilde{w}_n) = \frac{1}{\|w_n\|} F(w_n) \\ &= \frac{1}{\|w_n\|} F(x_n - z_n) = \frac{1}{\|w_n\|} F(x_n) \\ &= \frac{1}{\|w_n\|} y_n \rightarrow 0 \text{ as } n \rightarrow \infty \end{aligned}$$

because $\{y_n\}_{n \in \mathbb{N}}$ is bounded (being convergent) and $\|w_n\| \rightarrow \infty$ as $n \rightarrow \infty$. Since $\{\tilde{w}_n\}_{n \in \mathbb{N}}$ is bounded and T is compact, $\{T(\tilde{w}_n)\}_{n \in \mathbb{N}}$ has a convergent subsequence. Hence, passing to a subsequence if necessary, we may assume that $\{T(\tilde{w}_n)\}_{n \in \mathbb{N}}$ is convergent. As

$$\mu \tilde{w}_n = T(\tilde{w}_n) - \frac{1}{\|w_n\|} y_n \quad \text{for each } n \in \mathbb{N},$$

we get that $\{\tilde{w}_n\}_{n \in \mathbb{N}}$ is convergent, say to $\tilde{w} \in H$. Then $\|\tilde{w}\| = 1$ and

$$F(\tilde{w}) = \lim_{n \rightarrow \infty} F(\tilde{w}_n) = 0,$$

so $\tilde{w} \in \ker(F)$. But $\tilde{w}_n \in \ker F^\perp$ for all $n \in \mathbb{N}$, which is closed. Thus, we also have $\tilde{w} \in \ker(F)^\perp$. Hence, $\tilde{w} \in \ker(F) \cap \ker(F)^\perp = \{0\}$, i.e., $\tilde{w} = 0$, contradicting that $\|\tilde{w}\| = 1$. So we must have that $\{w_n\}_{n \in \mathbb{N}}$ is bounded, as claimed above.

Now, since T is compact, we get that the sequence $\{T(w_n)\}_{n \in \mathbb{N}}$ has a convergent subsequence. Passing again to a subsequence if necessary, we may assume that $\{T(w_n)\}_{n \in \mathbb{N}}$ is convergent. As

$$w_n = \frac{1}{\mu} (T(w_n) - (T - \mu I)(w_n)) = \frac{1}{\mu} (T(w_n) - F(w_n)) = \frac{1}{\mu} (T(w_n) - y_n)$$

for every $n \in \mathbb{N}$, this implies that $\{w_n\}_{n \in \mathbb{N}}$ is convergent, say to $w \in H$. Then we get

$$F(w) = \lim_{n \rightarrow \infty} F(w_n) = \lim_{n \rightarrow \infty} y_n = y,$$

i.e., $y \in F(H)$, as we wanted to show. ■

We will also need:

4.4. The Fredholm Alternative

Proposition 4.4.8. *Let H be a Hilbert space (over $\mathbb{K}$), $T \in \mathcal{K}(H)$ and $\mu \in \mathbb{K} \setminus \{0\}$. Then we have*

$$\dim(\ker(T - \mu I)) = \dim(\ker((T - \mu I)^*)) < \infty.$$

We note that it follows readily from Lemma 4.3.1 that

$$\dim(\ker(T - \mu I)) < \infty \quad \text{and} \quad \dim(\ker((T - \mu I)^*)) = \dim(\ker(T^* - \bar{\mu} I)) < \infty.$$

We divide the proof of the formula $\dim(\ker(T - \mu I)) = \dim(\ker((T - \mu I)^*))$ in several lemmas. We first show that it holds when $S \in \mathcal{F}(H)$.

Lemma 4.4.9. *Assume $S \in \mathcal{B}(H)$ has finite rank, i.e., $S \in \mathcal{F}(H)$. Then*

$$\dim(\ker(S - \mu I)) = \dim(\ker((S - \mu I)^*)).$$

Proof. Since S has finite rank, $S(H)$ is a finite-dimensional subspace of H . Moreover, since S^* then also has finite rank, $S^*(H)$ is finite-dimensional too. This readily implies that $M := S(H) + S^*(H)$ is a finite-dimensional subspace of H . Clearly, we have

$$(S - I)(M) \subseteq M \quad \text{and} \quad (S - I)^*(M) = (S^* - I)(M) \subseteq M.$$

Moreover, since $M^\perp \subseteq S^*(H)^\perp = \ker(S)$, we also have

$$(S - I)(\eta) = 0 + \eta = \eta \quad \text{for all } \eta \in M^\perp.$$

We can now define a linear map $L : M \rightarrow M$ by $L := (S - I)|_M$, i.e., $L(\xi) = S(\xi) - \xi$ for $\xi \in M$. We claim that

$$\begin{aligned} \ker(S - I) &= \ker(L) \\ \ker((S - I)^*) &= \ker(L^*) \end{aligned}$$

Indeed, assume first $\xi \in \ker(L)$. Then $(S - I)(\xi) = L(\xi) = 0$, so $\xi \in \ker(S - I)$. Conversely, assume $\xi \in \ker(S - I)$. Write $\xi = \xi_M + \eta$ with $\xi_M \in M$ and $\eta \in M^\perp$. Then we have

$$0 = (S - I)(\xi) = (S - I)(\xi_M) + (S - I)(\eta) = L(\xi_M) + \eta,$$

and $L(\xi_M) \in M$, $\eta \in M^\perp$. This implies that $L(\xi_M) = 0$ and $\eta = 0$, hence that

$$\xi = \xi_M + 0 = \xi_M \in M \quad \text{and} \quad L(\xi) = 0, \text{ i.e., } \xi \in \ker(L).$$

This shows that the first assertion in the claim above holds. The proof of the second assertion is similar and left to the reader.

4. On compact operators

As M is finite-dimensional, Example 4.4.5 tells us that

$$\dim(\ker(L)) = \dim(\ker(L^*)).$$

Thus, combining the claim above with this formula, we get that the desired equality holds. $\blacksquare$

Lemma 4.4.10. *Assume $B \in \mathcal{B}(H)$ and $\dim(\ker(B))$, $\dim(\ker(B^*)) < \infty$. Let $U \in \mathcal{B}(H)$ be invertible. Then we have*

$$\begin{aligned} \dim(\ker(UB)) &= \dim(\ker(B)), \\ \dim(\ker((UB)^*)) &= \dim(\ker(B^*)). \end{aligned}$$

Proof. For $x \in H$ we have

$$(UB)(x) = 0 \Leftrightarrow U(B(x)) = 0 \Leftrightarrow B(x) = 0 \text{ since } U \text{ is invertible.}$$

Thus, $\ker(UB) = \ker(B)$ and the first statement follows. Moreover, we have

$$(UB)^*(x) = 0 \Leftrightarrow B^*(U^*(x)) = 0 \Leftrightarrow U^*(x) \in \ker(B^*) \Leftrightarrow x \in (U^*)^{-1}(\ker(B^*))$$

(since U^* is invertible, with $(U^*)^{-1} = (U^{-1})^*$). This shows that $\ker((UB)^*) = (U^*)^{-1}(\ker(B^*))$, so we get that

$$\dim(\ker((UB)^*)) = \dim((U^*)^{-1}(\ker(B^*))) = \dim(\ker(B^*)),$$

proving the second statement. $\blacksquare$

Lemma 4.4.11. *Assume $T \in \mathcal{K}(H)$. Then*

$$\dim(\ker(T - I)) = \dim(\ker((T - I)^*)).$$

Proof. Since $\mathcal{F}(H)$ is dense in $\mathcal{K}(H)$, we can find $S \in \mathcal{F}(H)$ such that $\|T - S\| < 1$. Set $U := I - (T - S)$. Then $U \in \mathcal{B}(H)$ is invertible (with $U^{-1} = \sum_{n=0}^{\infty} (T - S)^n$), and we have

$$T - I = S + (T - S) - I = S - U = U(U^{-1}S - I) = U(\tilde{S} - I),$$

where $\tilde{S} := U^{-1}S \in \mathcal{F}(H)$. Lemma 4.4.9 gives that

$$\dim(\ker(\tilde{S} - I)) = \dim(\ker((\tilde{S} - I)^*)).$$

Moreover, applying Lemma 4.4.10, we get

$$\begin{aligned} \dim(\ker(T - I)) &= \dim(\ker(U(\tilde{S} - I))) = \dim(\ker(\tilde{S} - I)), \\ \dim(\ker((T - I)^*)) &= \dim(\ker((U(\tilde{S} - I))^*)) = \dim(\ker((\tilde{S} - I)^*)). \end{aligned}$$

Hence, the desired formula follows by combining the last three equalities. $\blacksquare$

Proof of Proposition 4.4.8. Since $\mu \neq 0$, we have

$$T - \mu I = \mu \left(\frac{1}{\mu} T - I \right) = \mu(\tilde{T} - I),$$

where $\tilde{T} := \frac{1}{\mu} T \in \mathcal{K}(H)$. Hence, using Lemma 4.4.11, we get

$$\begin{aligned} \dim(\ker(T - \mu I)) &= \dim(\ker(\mu(\tilde{T} - I))) = \dim(\ker(\tilde{T} - I)) \\ &= \dim(\ker((\tilde{T} - I)^*)) = \dim(\ker(\bar{\mu}(\tilde{T} - I)^*)) \\ &= \dim(\ker((\mu(\tilde{T} - I))^*)) = \dim(\ker((T - \mu I)^*)), \end{aligned}$$

as we wanted to show. ■

Theorem 4.4.12. Assume $T \in \mathcal{K}(H)$ and $\mu \in \mathbb{K} \setminus \{0\}$. Then $F := T - \mu I$ satisfies the Fredholm alternative.

Proof. Theorem 4.4.7 says that both F and F^* have closed ranges, while Proposition 4.4.8 says that $\dim(\ker(F^*)) = \dim(\ker(F)) < \infty$. Hence the conclusion follows from Proposition 4.4.4. ■

4.5 Exercises

Exercise 4.1. Let X, Y, Z denote normed spaces over $\mathbb{K}$. Consider $\lambda \in \mathbb{K}$, $T, T' \in \mathcal{B}(X, Y)$ and $S \in \mathcal{B}(Y, Z)$, so $ST \in \mathcal{B}(X, Z)$.

- a) Show that $\lambda T + T' \in \mathcal{K}(X, Y)$ if $T, T' \in \mathcal{K}(X, Y)$.
- b) Show that $ST \in \mathcal{K}(X, Z)$ if $T \in \mathcal{K}(X, Y)$.
- c) Show that $ST \in \mathcal{K}(X, Z)$ if $S \in \mathcal{K}(Y, Z)$.
- d) Set $\mathcal{K}(X) = \mathcal{K}(X, X)$. Deduce that $ST \in \mathcal{K}(X)$ if $S \in \mathcal{B}(X)$ and $T \in \mathcal{K}(X)$, or if $S \in \mathcal{K}(X)$ and $T \in \mathcal{B}(X)$.

Exercise 4.2. Let $X = \ell^p(\mathbb{N})$, $\lambda \in \ell^\infty(\mathbb{N})$, and $M_\lambda \in \mathcal{B}(X)$ be the associated multiplication operator, cf. Example 4.1.8.

Show that $\lambda \in c_0(\mathbb{N})$ if M_λ is compact.

(It therefore follows that M_λ is compact if and only if $\lambda \in c_0(\mathbb{N})$.)

Exercise 4.3. Let X be a normed space, H be a Hilbert space, and let $T \in \mathcal{K}(X, H)$. Show that $\overline{T(X)}$ is separable.

Exercise 4.4. Let H be an infinite-dimensional Hilbert space and let $T \in \mathcal{K}(H)$. Show that $\langle T(u_n), u_n \rangle \rightarrow 0$ as $n \rightarrow \infty$ whenever $\{u_n\}_{n \in \mathbb{N}}$ is an orthonormal sequence in H .

4. On compact operators

Exercise 4.5. Let H be a Hilbert space and assume $P \in \mathcal{B}(H)$ satisfies $P^2 = P$.

Show that P has finite-rank if (and only if) P is compact.

Exercise 4.6. Let H be a separable Hilbert space, $H \neq \{0\}$.

a) Show that $\mathcal{F}(H) \subseteq \mathcal{HS}(H)$, and that $\mathcal{F}(H)$ is dense in $\mathcal{HS}(H)$ w.r.t. $\|\cdot\|_2$.

b) Assume that $T \in \mathcal{HS}(H)$ and $S \in \mathcal{B}(H)$. Show that both ST and TS belong to $\mathcal{HS}(H)$, and that we have

$$\|ST\|_2 \leq \|S\| \|T\|_2, \quad \|TS\| \leq \|T\|_2 \|S\|.$$

c) Let $\mathcal{B} = \{u_j\}_{j \in N}$ be an orthonormal basis for H , where $N = \{1, \dots, n\}$ if $\dim(H) = n < \infty$, while $N = \mathbb{N}$ otherwise.

For $S, T \in \mathcal{HS}(H)$, set

$$\langle S, T \rangle_2 := \sum_{j \in N} \langle S(u_j), T(u_j) \rangle.$$

Show that this gives a well-defined inner product on $\mathcal{HS}(H)$, and check that the associated norm is the Hilbert-Schmidt norm $\|\cdot\|_2$.

d) Show that $\mathcal{HS}(H)$ is complete w.r.t. $\|\cdot\|_2$, so that $\mathcal{HS}(H)$ is a Hilbert space w.r.t. the inner product from c).

Exercise 4.7. Let $H = L^2(\mathbb{R}, \mathcal{A}, \mu)$ where $\mathcal{A}$ denote all Lebesgue measurable subsets of $\mathbb{R}$ and μ is the Lebesgue measure. For which $f \in L^\infty(\mathbb{R}, \mathcal{A}, \mu)$ is the multiplication operator $M_f \in \mathcal{B}(H)$ compact?

Exercise 4.8. Let H be a Hilbert space, $T \in \mathcal{K}(H)$ and $\lambda \in \mathbb{K}, \lambda \neq 0$. Assume that there exists a sequence $\{x_n\}_{n \in \mathbb{N}}$ of unit vectors in H such that $\|T(x_n) - \lambda x_n\| \rightarrow 0$ as $n \rightarrow \infty$. Show that λ is an eigenvalue of T .

Exercise 4.9. Consider $H = L^2([-\pi, \pi])$ (with respect to the normalized Lebesgue measure). Let $g \in C([-\pi, \pi])$ be periodic, i.e., satisfies that $g(-\pi) = g(\pi)$, and extend g to a periodic function $\tilde{g}$ on $\mathbb{R}$ with period 2π . Define $G : [-\pi, \pi] \times [-\pi, \pi] \rightarrow \mathbb{C}$ by

$$G(s, t) = \tilde{g}(s - t).$$

a) Check that G is continuous, so that the associated integral operator T_G belongs to $\mathcal{HS}(H)$ (hence is compact).

c) Decide when T_G is self-adjoint.

b) Let $k \in \mathbb{Z}$ and recall that $e_k(t) = e^{ikt}$ for all $t \in [-\pi, \pi]$. Check that e_k is an eigenvector for the operator T_G . Deduce that T_G is diagonalizable (with respect to $\{e_k\}_{k \in \mathbb{Z}}$).

c) Show that $\|T_G\|_2 = \|g\|_2 = \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} |g(t)|^2 dt \right)^{1/2}$.

Exercise 4.10. Consider $H = L^2([0, 1])$ (with respect to Lebesgue measure) and the integral operator $T_K \in \mathcal{B}(H)$ associated to the kernel given by

$$K(s, t) = \min(s, t)$$

for all (s, t) in $[0, 1] \times [0, 1]$, cf. Example 4.2.11.

a) Explain why T_K is self-adjoint and compact. Then check that the set $\mathcal{U} := \{u_n : n \in \mathbb{N}\}$, where

$$u_n(t) := \sqrt{2} \sin \left(\left(n - \frac{1}{2} \right) \pi t \right) \quad \text{for all } t \in [0, 1], n \in \mathbb{N},$$

is an orthonormal set of eigenvectors for T_K .

b) It can be shown that $\mathcal{U}$ is an orthonormal basis for H . Is it possible to deduce this from a) and the spectral theorem for T_K ?

Exercise 4.11. Let $S, T \in \mathcal{B}(H)$.

a) Assume there exists an orthonormal basis for H whose elements are eigenvectors for both S and T . Check that S commutes with T .

b) Assume S and T are compact and self-adjoint, and that S commutes with T . Show that there exists an orthonormal basis for H whose elements are eigenvectors for both S and T .

Hint: Start by considering an eigenvalue λ of T and study how S acts on the corresponding eigenspace E_λ^T .

Exercise 4.12. Assume H is a Hilbert space over $\mathbb{R}$, and let $T \in \mathcal{B}(H)$.

a) Assume that T is diagonalizable (as defined in Remark 4.3.6). Check that T is self-adjoint.

b) Let T be compact. Deduce that T is diagonalizable if and only if T is self-adjoint.

Exercise 4.13. Assume H is a Hilbert space over $\mathbb{C}$, and let $T \in \mathcal{B}(H)$.

a) Assume that T is diagonalizable (as defined in Remark 4.3.6). Check that T is normal.

4. On compact operators

b) Show that T is normal if and only if $\operatorname{Re}(T)$ and $\operatorname{Im}(T)$ commutes with each other.

c) Let T be compact. Show that T is diagonalizable if and only if T is normal.

Hint: The implication $(\Rightarrow)$ follows from a). For $(\Leftarrow)$, use b) and Exercise 4.11 b).

Exercise 4.14. Let H be an infinite-dimensional separable Hilbert space and assume $\mathcal{B}$ is an orthonormal basis for H , enumerated as $\mathcal{B} = \{v_j\}_{j \in \mathbb{N}}$. Let $\{\mu_j\}_{j \in \mathbb{N}}$ be a bounded sequence in $\mathbb{K}$ and let $D \in \mathcal{B}(H)$ denote the associated diagonal operator (w.r.t. $\mathcal{B}$).

a) Show that D is compact if and only if $\lim_{j \rightarrow \infty} \mu_j = 0$.

(Note: If you have looked at Example 4.1.8 and solved Exercise 4.2, this should not be difficult).

b) Show that D is Hilbert-Schmidt if and only if $\{\mu_j\}_{j \in \mathbb{N}} \in \ell^2(\mathbb{N})$, in which case we have

$$\|D\|_2 = \left(\sum_{j=1}^{\infty} |\mu_j|^2 \right)^{1/2}.$$

Exercise 4.15. Let H be an infinite-dimensional separable Hilbert space and let $T \in \mathcal{K}(H)$ be selfadjoint, $T \neq 0$. Assume that you have found an orthonormal basis $\mathcal{B} = \{v_j\}_{j \in \mathbb{N}}$ for H consisting of eigenvectors for T , and let $\mu_j \in \mathbb{R}$ denote the eigenvalue of T corresponding to each v_j .

a) Show that the sequence $\{\mu_j\}_{j \in \mathbb{N}}$ is bounded, hence that T is the diagonal operator associated with this sequence (w.r.t. $\mathcal{B}$). Deduce from the previous exercise that $\lim_{j \rightarrow \infty} \mu_j = 0$.

b) As in the spectral theorem, set

$$L := \{\lambda \in \mathbb{R} \mid \lambda \text{ is a nonzero eigenvalue of } T\}.$$

Set also

$$\tilde{L} := \{\lambda \in \mathbb{R} \mid \lambda \text{ is an eigenvalue of } T\},$$

so $L = \tilde{L} \setminus \{0\}$. Show the following assertions:

(i) $\tilde{L} = \{\mu_j \mid j \in \mathbb{N}\}$ and $L = \{\mu_j \mid j \in \mathbb{N}, \mu_j \neq 0\}$.

(ii) If $\lambda \in L$ and $N_\lambda := \{j \in \mathbb{N} \mid \mu_j = \lambda\}$, then N_λ is a finite subset of $\mathbb{N}$ and $\{v_j \mid j \in N_\lambda\}$ is an o.n.b. for E_λ .

(iii) If $\mu_j \neq 0$ for all $j \in \mathbb{N}$, then $\ker(T) = \{0\}$.

(iv) If $N_0 := \{j \in \mathbb{N} \mid \mu_j = 0\}$ is nonempty, then $\{v_j \mid j \in N_0\}$ is an o.n.b. for $\ker(T)$.

Exercise 4.16. Let $H = L^2([0, 1])$ (with usual Lebesgue measure) and let $T = M_f$ be the self-adjoint operator in $\mathcal{B}(H)$ given by multiplication with the function $f(t) = t$ on $[0, 1]$, cf. Example 2.5.4.

Show that $T(H)$ is not closed, i.e., that T does not have closed range. Show also that T is not compact.

Exercise 4.17. Let $H = \ell^2(\mathbb{N})$, let $\lambda \in \ell^\infty(\mathbb{N})$ be given by $\lambda(n) = \frac{1}{n}$ for all $n \in \mathbb{N}$, and let $T = M_\lambda \in \mathcal{B}(H)$ denote the associated multiplication operator. Note that T is compact, as follows from Example 4.1.8.

Show that $\overline{T(H)} = H$ and $T(H) \neq H$, so T does not have closed range.

Exercise 4.18. Let H be a Hilbert space and $T \in \mathcal{B}(H)$. Let us say that T is *bounded from below* if there exists some $\alpha > 0$ such that $\alpha \|x\| \leq \|T(x)\|$ for all $x \in H$. For example, T is bounded from below when T is an isometry.

Show that if T is bounded from below, then T has closed range.

Exercise 4.19. Let H be a Hilbert space and assume $F \in \mathcal{B}(H)$ satisfies the Fredholm alternative. Deduce that F is injective if and only if F is surjective.

Exercise 4.20. Let H be a Hilbert space, and let $T \in \mathcal{K}(H)$ be self-adjoint. Assume $\mu \in \mathbb{K}$, $\mu \neq 0$, is *not* an eigenvalue of T , i.e. $T - \mu I$ is injective.

Let $y \in H$, let $\mathcal{E}' = \{u_j\}_{j \in J}$ be an enumeration of the orthonormal basis for $M = \overline{T(H)}$ obtained in the spectral theorem for T , and let $\rho_j \neq 0$ denote the eigenvalue of T corresponding to u_j (as in the proof of Corollary 4.3.5).

a) Show that the series

$$\sum_{j \in J} \frac{\langle y, u_j \rangle}{\rho_j - \mu} u_j$$

converges to some $h \in H$.

b) Set $z := y - P_M(y)$ and $x := h - \frac{1}{\mu} z$. Show that $(T - \mu I)(x) = y$.

c) Deduce that $F := T - \mu I$ is surjective (hence that it is bijective). [This shows that F satisfies a) in the Fredholm alternative.]

Exercise 4.21. In the proof of Theorem 4.4.6, check that the sequence $\{c_j\}_{j \in J}$ constructed towards the end of the first part (under the assumption that y is orthogonal to E_μ^T) belongs to $\ell^2(J)$.

CHAPTER 5

An introduction to Sturm-Liouville theory

The purpose of this chapter is to illustrate how the spectral theorem for a compact self-adjoint operator on a Hilbert space may be used to study some classical Sturm-Liouville problems. For simplicity we will only discuss the so-called regular case.

5.1 Regular Sturm-Liouville systems

When $[a, b] \subset \mathbb{R}$ and $n \in \mathbb{N}$, we let $C^n([a, b])$ denote the space of n times continuously differentiable complex functions on $[a, b]$. A *regular Sturm-Liouville system* on $[a, b]$ is a second order linear differential equation of the form

$$-(py')' + qy = \lambda \rho y, \quad (5.1.1)$$

where

- $p \in C^1([a, b])$ is real-valued and $p(x) \neq 0$ for all $x \in [a, b]$,
- $q, \rho \in C([a, b])$ are real-valued and $\rho(x) \neq 0$ for all $x \in [a, b]$,
- $\lambda \in \mathbb{C}$,

and the unknown function $y = y(x)$, which necessarily has to lie in $C^2([a, b])$, is required to satisfy boundary conditions of the type

$$\alpha_1 y(a) + \alpha_2 y'(a) = 0, \quad \beta_1 y(b) + \beta_2 y'(b) = 0, \quad (5.1.2)$$

for some $(\alpha_1, \alpha_2), (\beta_1, \beta_2) \in \mathbb{R}^2 \setminus \{(0, 0)\}$.

5. An introduction to Sturm-Liouville theory

Ideally, the Sturm-Liouville problem is to determine the values of λ for which there exist non-trivial solutions of equation (5.1.1) satisfying the conditions (5.1.2), and to describe these solutions. These values of λ are called the *eigenvalues* of the system, and the corresponding solutions y are called *eigenfunctions* of the system. A concrete answer to this problem is not possible in general, but one may still obtain some valuable theoretical information about it.

Since we only intend to give a small taste of Sturm-Liouville theory, we will assume that $p(x) = \rho(x) = 1$ for all $x \in [a, b]$, in which case equation (5.1.1) simplifies to

$$-y'' + qy = \lambda y. \quad (5.1.3)$$

A suitably scaled version of this equation appears for example as the one dimensional time-independent Schrödinger equation in quantum mechanics (where it is usually considered on the whole real line).

Set $Y = \{y \in C^2([a, b]) : y \text{ satisfies the boundary conditions (5.1.2)}\}$. We will consider Y and $C([a, b])$ as inner product spaces w.r.t. to the inner product given by

$$\langle f, g \rangle = \int_a^b f(x) \overline{g(x)} dx.$$

Letting $D : Y \rightarrow C([a, b])$ be the linear operator defined by

$$D(y) = -y'' + qy,$$

it is clear that our Sturm-Liouville system may be written as

$$D(y) = \lambda y \quad \text{where } y \in Y. \quad (5.1.4)$$

Although the associated Sturm-Liouville problem looks like a familiar eigenvalue/eigenvector problem, it is not obvious how to proceed. The fact that Y and $C([a, b])$ are not Hilbert spaces (as they are not complete) can easily be fixed because both spaces can be considered as dense subspaces of $L^2([a, b])$. However, the operator D is not bounded (check this!), so it does not extend to a bounded operator on $L^2([a, b])$. We will have to work quite a bit to recast the problem into one involving a compact selfadjoint operator.

- We will first study the second order differential equation

$$-y'' + qy = \lambda y \quad \text{with } y \in C^2([a, b]) \quad (5.1.5)$$

5.2. A second order differential equation

and show that its solution space

$$S_\lambda := \{y \in C^2([a, b]) : -y'' + qy = \lambda y\}$$

is 2-dimensional for every $\lambda \in \mathbb{C}$.¹ Note that trying to find out when there exists some $y \in S_\lambda \setminus \{0\}$ which also belongs to Y , which would solve our problem, is not possible because a concrete description of S_λ is not available in general.

- Next, we will establish some spectral properties of the operator D .
- Thirdly, we will study the case where D is 1-1. We will then show that D maps Y bijectively onto $C([a, b])$, and that there exists a compact self-adjoint integral operator $T_G : L^2([a, b]) \rightarrow L^2([a, b])$ such that its restriction to $C([a, b])$ is the inverse of D . Applying the spectral theorem to T_G will enable us to describe how the solution to our Sturm-Liouville problem looks like in this case.
- Finally, we will explain how we can handle the general case where D is not assumed to be 1-1.

5.2 A second order differential equation

We recall that $\lambda \in \mathbb{C}$. In this section it is not important that the function $q \in C([a, b])$ is assumed to be real-valued.

Theorem 5.2.1. *Let $c \in [a, b]$ and $z_1, z_2 \in \mathbb{C}$. Then there exists a unique function $y \in C^2([a, b])$ satisfying that*

$$-y'' + qy = \lambda y \quad \text{and} \quad y(c) = z_1, y'(c) = z_2. \quad (5.2.1)$$

Proof. Suppose first that $y \in C^2([a, b])$ satisfies (5.2.1), that is,

$$y'' = (q - \lambda)y, \quad y(c) = z_1 \quad \text{and} \quad y'(c) = z_2. \quad (5.2.2)$$

For every $u \in [a, b]$, we get

$$y'(u) - z_2 = y'(u) - y'(c) = \int_c^u y''(t) dt = \int_c^u (q(t) - \lambda) y(t) dt.$$

¹This fact holds for the solution space of any homogeneous second order linear ordinary differential equation, as some students may have seen in a previous course. We will give a self-contained proof in our case.

5. An introduction to Sturm-Liouville theory

This gives that

$$\begin{aligned}
 y(x) - z_1 - z_2(x - c) &= y(x) - y(c) - z_2(x - c) = \int_c^x (y'(u) - z_2) du \\
 &= \int_c^x \int_c^u (q(t) - \lambda) y(t) dt du \\
 &= \int_c^x \int_t^x (q(t) - \lambda) y(t) du dt \\
 &= \int_c^x (x - t)(q(t) - \lambda) y(t) dt
 \end{aligned}$$

for all $x \in [a, b]$, hence that y satisfies that the integral equation

$$y(x) = z_1 + z_2(x - c) + \int_c^x (x - t)(q(t) - \lambda) y(t) dt \quad \text{for all } x \in [a, b] \quad (5.2.3)$$

Conversely, if $y \in C([a, b])$ satisfies (5.2.3), then it is an easy exercise to check that y belongs to $C^2([a, b])$ and satisfies (5.2.2).

Now, let $T : C([a, b]) \rightarrow C([a, b])$ be the operator defined for each f in $C([a, b])$ by

$$[T(f)](x) = z_1 + z_2(x - c) + \int_c^x (x - t)(q(t) - \lambda) f(t) dt$$

for all $x \in [a, b]$.

Let us consider $C([a, b])$ as a complete metric space w.r.t. the metric $d(f, g) := \|f - g\|_\infty = \sup\{|f(x) - g(x)| : x \in [a, b]\}$ and set

$$K := \sup\{|(x - t)(q(t) - \lambda)| : x, t \in [a, b]\} < \infty.$$

Let $f, g \in C([a, b])$ and $x \in [a, b]$. By induction on $n \in \mathbb{N}$, one easily shows that

$$\left| [T^n(f) - T^n(g)](x) \right| \leq \frac{1}{n!} K^n |x - c|^n \|f - g\|_\infty.$$

This implies that

$$\|T^n(f) - T^n(g)\|_\infty \leq \frac{K^n (b - a)^n}{n!} \|f - g\|_\infty.$$

It follows that T^n is a contraction when n is so large that $\frac{K^n (b-a)^n}{n!} < 1$. Hence, Banach's fixed point theorem ² gives that T has a unique fixed point, say y , in $C([a, b])$. This means that y is the unique function in $C([a, b])$ such that $y = T(y)$, i.e., such that y satisfies (5.2.3). Taking into account what we proved in the first part of the proof, we are done. ■

²cf. Lindstrøm's book *Spaces*, Exercise 3.4.7.

5.2. A second order differential equation

Corollary 5.2.2. *Let $\lambda \in \mathbb{C}$, $c \in [a, b]$, and recall that*

$$S_\lambda = \{y \in C^2([a, b]) : -y'' + qy = \lambda y\}.$$

Then the map $T_{\lambda, c} : S_\lambda \rightarrow \mathbb{C}^2$ defined by $T_{\lambda, c}(y) = (y(c), y'(c))$ for every $y \in S_\lambda$ is an isomorphism. Hence, $\dim(S_\lambda) = 2$.

Proof. Theorem 5.2.1 shows that the map $T_{\lambda, c}$ is 1-1 and onto. As it is obviously linear, it is an isomorphism. Hence, $\dim(S_\lambda) = \dim(\mathbb{C}^2) = 2$. ■

Remark 5.2.3. It should be noted that Theorem 5.2.1 is essentially an existence result (although our method of proof gives a way to approximate the unique solution of (5.2.1) by picking some $y_0 \in C([a, b])$ and computing $T^n(y_0)$ for large enough n). Explicit formulas for a basis of S_λ are only known when q is a constant function. To illustrate Corollary 5.2.2, we recall how these bases look like in this case.

Assume $q(x) = \omega$ for all $x \in [a, b]$ for some $\omega \in \mathbb{C}$. Then $-y'' + qy = \lambda y$ can be rewritten as the second order homogeneous equation $y'' + (\lambda - \omega)y = 0$, which we know can be solved by considering the characteristic equation $z^2 + (\lambda - \omega) = 0$:

If $\lambda \neq \omega$, then, letting $(\omega - \lambda)^{1/2}$ denote a square root of $\omega - \lambda$ in $\mathbb{C}$, we get that S_λ consists of the functions of the form

$$y(x) = C_1 e^{(\omega - \lambda)^{1/2}x} + C_2 e^{-(\omega - \lambda)^{1/2}x}, \quad x \in [a, b],$$

where $C_1, C_2 \in \mathbb{C}$. Thus $\{e^{(\omega - \lambda)^{1/2}x}, e^{-(\omega - \lambda)^{1/2}x}\}$ is a basis for S_λ in this case.

If $\lambda = \omega$, the equation is $y'' = 0$ and $\{1, x\}$ is obviously a basis for S_λ . □

Remark 5.2.4. Suppose that q is real-valued, $\lambda \in \mathbb{R}$ and $y \in S_\lambda$. Then it is not difficult to verify that $\bar{y} \in S_\lambda$, so that $\operatorname{Re}(y)$ and $\operatorname{Im}(y)$ also lie in S_λ . Moreover, if it happens that $y(c)$ and $y'(c)$ are both real numbers for some $c \in [a, b]$, then the function y has to be real-valued: indeed, we then have $(\operatorname{Im}(y))(c) = 0 = (\operatorname{Im}(y))'(c)$, so Theorem 5.2.1 implies that $\operatorname{Im}(y)$ is the zero function on $[a, b]$. □

Remark 5.2.5. Consider $y_1, y_2 \in C^2([a, b])$. Define $W_{y_1, y_2} \in C^1([a, b])$ by

$$W_{y_1, y_2}(x) := \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} = y_1(x)y_2'(x) - y_1'(x)y_2(x) \text{ for each } x \in [a, b].$$

$W_{y_1, y_2}(x)$ is called the *Wronsky determinant* of (y_1, y_2) at x .

5. An introduction to Sturm-Liouville theory

Assume that $y_1, y_2 \in S_\lambda$ and let $c \in [a, b]$. Corollary 5.2.2 implies that the set $\{y_1, y_2\}$ is a basis for S_λ if and only if the vectors $(y_1(c), y_1'(c)), (y_2(c), y_2'(c))$ are linearly independent in $\mathbb{C}^2$, i.e., $W_{y_1, y_2}(c) \neq 0$. Hence we get that if $\{y_1, y_2\}$ is a basis for S_λ , then $W_{y_1, y_2}(x) \neq 0$ for all $x \in [a, b]$.

The Wronsky determinant appears in the following lemma (sometimes called *Lagrange's lemma*), which will be useful to us later:

Lemma 5.2.6. *Consider the operator $\widetilde{D} : C^2([a, b]) \rightarrow C([a, b])$ given by*

$$\widetilde{D}(y) = -y'' + qy \quad \text{for every } y \in C^2([a, b]).$$

Let $f, g \in C^2([a, b])$. Then the following identity holds:

$$\widetilde{D}(f)g - \widetilde{D}(g)f = (W_{f,g})'.$$

Proof. We have

$$\begin{aligned} (W_{f,g})' &= (fg' - gf')' = fg'' + f'g' - gf'' - g'f' = fg'' - gf'' \\ &= (-f''g + qfg) - (g''f - qgf) = \widetilde{D}(f)g - \widetilde{D}(g)f. \end{aligned}$$

■

5.3 Some spectral properties

Let $(\alpha_1, \alpha_2), (\beta_1, \beta_2) \in \mathbb{R}^2 \setminus \{(0, 0)\}$ and $q \in C([a, b])$ be real-valued. We recall that

$$Y = \{y \in C^2([a, b]) \mid y \text{ satisfies the boundary conditions (5.3.1)}\},$$

where

$$\alpha_1 y(a) + \alpha_2 y'(a) = 0, \quad \beta_1 y(b) + \beta_2 y'(b) = 0, \quad (5.3.1)$$

and $D : Y \rightarrow C[a, b]$ is the operator defined by $D(y) = -y'' + qy$ for $y \in Y$.

Let $\lambda \in \mathbb{C}$ and set $E_\lambda := \{y \in Y : D(y) = \lambda y\}$. We will say that λ is an *eigenvalue* of D if the subspace E_λ is non-trivial, in which case E_λ is called the *eigenspace* of D associated to λ . We note that $E_\lambda \subseteq S_\lambda$, so Corollary 5.2.2 implies that $\dim(E_\lambda) \leq 2$.

Proposition 5.3.1. *Let $f, g \in Y$. Then we have*

$$i) \quad D(f)g - D(g)f = (fg' - gf')',$$

$$ii) \quad \langle D(f), g \rangle = \langle f, D(g) \rangle.$$

5.3. Some spectral properties

Proof. *i)* Since $D = (\widetilde{D})|_Y$, this identity follows from Lagrange's lemma (Lemma 5.2.6).

ii) It is easy to check that $\bar{g} \in Y$ and $\overline{D(\bar{g})} = D(\bar{g})$. Thus, using *i)*, we get

$$\begin{aligned} \langle D(f), g \rangle - \langle f, D(g) \rangle &= \int_a^b [D(f)\bar{g} - fD(\bar{g})](t) dt \\ &= \int_a^b (f\bar{g}' - \bar{g}f')(t) dt = \left[(f\bar{g}' - \bar{g}f')(t) \right]_a^b \\ &= f(b)\bar{g}'(b) - \bar{g}(b)f'(b) - f(a)\bar{g}'(a) + \bar{g}(a)f'(a). \end{aligned}$$

Now, since f and $\bar{g}$ both satisfy 5.3.1, we have

$$\begin{bmatrix} f(b) & f'(b) \\ \bar{g}(b) & \bar{g}'(b) \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Since $(\beta_1, \beta_2) \neq (0, 0)$, this implies that $f(b)\bar{g}'(b) - \bar{g}(b)f'(b) = 0$. A similar argument gives that $f(a)\bar{g}'(a) - \bar{g}(a)f'(a) = 0$. Inserting these two equalities in the computation above, we get that

$$\langle D(f), g \rangle - \langle f, D(g) \rangle = 0,$$

as desired. ■

Part *ii)* of Proposition 5.3.1 shows that the operator D enjoys a property similar to self-adjointness. Proceeding exactly as we did for bounded self-adjoint operators on Hilbert spaces, one deduces readily that the following result holds.

Corollary 5.3.2. *All possible eigenvalues of D are real, and the associated eigenspaces are orthogonal to each other.*

Note that we don't know yet whether D has some eigenvalues. Anyhow, we can say more about its possible eigenspaces.

Proposition 5.3.3. *All possible eigenspaces of D are one-dimensional.*

Proof. Let $\lambda \in \mathbb{C}$. Recall that $\widetilde{D} : C^2([a, b]) \rightarrow C([a, b])$ is defined by $\widetilde{D}(y) = -y'' + qy$ for $y \in C^2([a, b])$, so $D = (\widetilde{D})|_Y$. We first consider the space

$$L_\lambda := \{y \in C^2([a, b]) : \widetilde{D}(y) = \lambda y \text{ and } y \text{ satisfies (5.3.2)}\},$$

where

$$\alpha_1 y(a) + \alpha_2 y'(a) = 0. \tag{5.3.2}$$

5. An introduction to Sturm-Liouville theory

Note that the condition (5.3.2) says that the vector $(y(a), y'(a))$ belongs to $M := \text{Span}\{(-\alpha_2, \alpha_1)\}$. Now, Corollary 5.2.2 (with $c = a$) gives that $L_\lambda = T_{\lambda,a}^{-1}(M)$. Since $\dim(M) = 1$ and $T_{\lambda,a}$ is an isomorphism, we get that L_λ is a one-dimensional subspace of S_λ .

By a similar argument, we get that

$$R_\lambda := \{y \in C^2([a, b]) : \widetilde{D}(y) = \lambda y \text{ and } y \text{ satisfies (5.3.3)}\},$$

where

$$\beta_1 y(b) + \beta_2 y'(b) = 0, \quad (5.3.3)$$

is a one-dimensional subspace of S_λ .

Clearly, we have $E_\lambda = L_\lambda \cap R_\lambda$. So there are only two possibilities: either $E_\lambda = \{0\}$ or $\dim(E_\lambda) = 1$. Hence, if λ is an eigenvalue of D , we must have $\dim(E_\lambda) = 1$. ■

Remark 5.3.4. We use the notation introduced in the proof above. Assume that $\lambda \in \mathbb{C}$ is *not* an eigenvalue of D , and pick $u \in L_\lambda$, $u \neq 0$, $v \in R_\lambda$, $v \neq 0$. It is clear from this proof that $L_\lambda = \text{Span}\{u\}$ and $R_\lambda = \text{Span}\{v\}$. As $L_\lambda \cap R_\lambda = E_\lambda = \{0\}$, the vectors u and v must be linearly independent. Since they both lie in S_λ , which is 2-dimensional by Corollary 5.2.2, we can then conclude that $\{u, v\}$ is a basis for S_λ . □

5.4 A special case

We return to the Sturm-Liouville problem for the equation $D(y) = \lambda y$, $y \in Y$. As long as we are not able to show that D has eigenvalues, it is not possible for us to make efficient use of its spectral properties. Ideally, we would like to show that D is diagonalizable, in the sense that there exists a sequence of eigenfunctions of D in Y which forms an orthonormal basis for $L^2([a, b])$. It appears to be a good idea to first turn our attention to the inverse of D , whenever it exists.

We therefore assume throughout this section that $D : Y \rightarrow C([a, b])$ is 1-1.

We will see how to get rid of this assumption in the next section.

Using the assumption that D is 1-1, we will show that $D(Y) = C([a, b])$, i.e., D is onto, and that the inverse operator

$$D^{-1} : C([a, b]) \rightarrow Y$$

is an integral operator associated to a continuous kernel $G : [a, b] \times [a, b] \rightarrow \mathbb{R}$.

Given a function $f \in C([a, b])$, it should be noted here that the standard way of showing that the differential equation $-y'' + qy = f$ has a solution is to pick a basis for the associated homogeneous equation and use the method called *variation of parameters*. We will not discuss this method here because we can follow a shorter path.

Since the operator D is linear, the fact that D is 1-1 means that its kernel is trivial, that is, 0 is *not* an eigenvalue of D . As we saw in Remark 5.3.4, we can then pick a basis $\{u, v\}$ for

$$S_0 = \{y \in C^2([a, b]) : \widetilde{D}(y) = 0\} = \{y \in C^2([a, b]) : y'' = qy\}$$

such that

- u satisfies the condition $\alpha_1 u(a) + \alpha_2 u'(a) = 0$,
- v satisfies the condition $\beta_1 v(b) + \beta_2 v'(b) = 0$.

Since q is real-valued, we can also assume that u and v are real-valued, cf. Remark 5.2.4.

We note that Remark 5.2.5 tells us that $W_{u,v}(x) \neq 0$ for all $x \in [a, b]$. Moreover, as $\widetilde{D}(u) = \widetilde{D}(v) = 0$, Lemma 5.2.6 gives that

$$(W_{u,v})' = \widetilde{D}(u)v - \widetilde{D}(v)u = 0.$$

Hence, $W_{u,v}$ is a constant function on $[a, b]$. This means that

$$W_{u,v}(x) = u(x)v'(x) - v(x)u'(x) = W \quad \text{for all } x \in [a, b]$$

for some $W \in \mathbb{R} \setminus \{0\}$.

We can now define the associated *Green's kernel* $G : [a, b] \times [a, b] \rightarrow \mathbb{R}$ by

$$G(x, t) := -\frac{1}{W} \cdot \begin{cases} u(x)v(t) & \text{if } a \leq x \leq t \leq b, \\ u(t)v(x) & \text{if } a \leq t \leq x \leq b. \end{cases}$$

It is then straightforward to see that G is continuous. Hence we may form the associated integral operator $T_G : L^2([a, b]) \rightarrow L^2([a, b])$, which satisfies that

$$[T_G(f)](x) = \int_a^b G(x, t) f(t) dt$$

for all $f \in C([a, b])$ and $x \in [a, b]$. We note that T_G maps $C([a, b])$ into Y :

5. An introduction to Sturm-Liouville theory

Proposition 5.4.1. *Let $f \in C([a, b])$ and set $y := T_G(f)$. Then $y \in Y$. Moreover, $D(y) = f$.*

Proof. Let $x \in [a, b]$. Using the definitions of G and T_G we get

$$y(x) = - \int_a^x W^{-1} v(x) u(t) f(t) dt - \int_x^b W^{-1} u(x) v(t) f(t) dt.$$

This implies that $-Wy(x) = v(x)A(x) + u(x)B(x)$, where

$$A(x) := \int_a^x u(t) f(t) dt \quad \text{and} \quad B(x) := \int_x^b v(t) f(t) dt.$$

Thus we get

$$\begin{aligned} -Wy'(x) &= v'(x)A(x) + v(x)A'(x) + u'(x)B(x) + u(x)B'(x) \\ &= v'(x)A(x) + v(x)u(x)f(x) + u'(x)B(x) - u(x)v(x)f(x) \\ &= v'(x)A(x) + u'(x)B(x). \end{aligned}$$

Since v', A, u' and B all lie in $C^1([a, b])$, we see that $y \in C^2([a, b])$.

Further, using that $A(a) = 0$ and $\alpha_1 u(a) + \alpha_2 u'(a) = 0$, we get

$$\begin{aligned} \alpha_1 y(a) + \alpha_2 y'(a) &= \frac{-1}{W} \left(\alpha_1 v(a)A(a) + \alpha_1 u(a)B(a) + \alpha_2 v'(a)A(a) + \alpha_2 u'(a)B(a) \right) \\ &= \frac{-1}{W} \left(\alpha_1 u(a) + \alpha_2 u'(a) \right) B(a) = 0. \end{aligned}$$

In a similar way we get $\beta_1 v(b) + \beta_2 v'(b) = 0$. Thus we have shown that $y \in Y$.

To verify the second assertion, we first compute $-Wy''$. Since $u'' = qu$, $v'' = qv$, and $v'u - u'v = W$ on $[a, b]$, we get

$$\begin{aligned} -Wy'' &= (v'A + u'B)' = v''A + v'A' + u''B + u'B' \\ &= q(vA + uB) + (v'u - u'v)f \\ &= -qWy + (v'u - u'v)f \\ &= W(f - qy). \end{aligned}$$

Thus, $-y'' = f - qy$, which gives

$$D(y) = -y'' + qy = f - qy + qy = f,$$

as desired. ■

5.4. A special case

The first part of Proposition 5.4.1 shows that D is onto $C([a, b])$. Since D is also 1-1 (by assumption), D has an inverse map $D^{-1} : C([a, b]) \rightarrow Y$. By definition, if $f \in C([a, b])$, then

$$D^{-1}(f) := y ,$$

where $y \in Y$ is the unique function in Y such that $D(y) = f$.

Thus we see that the second part of Proposition 5.4.1 tells us that

$$D^{-1}(f) = T_G(f) \quad \text{for every } f \in C([a, b]), \text{ i.e., } D^{-1} = (T_G)|_{C([a, b])}.$$

Since G is real-valued and $G(x, t) = G(t, x)$ for all (x, t) in $[a, b] \times [a, b]$ (check!), we get that T_G is self-adjoint. As T_G is also compact (indeed, it is a Hilbert-Schmidt operator on $L^2([a, b])$, cf. Example 4.2.11), we are in the position to apply the spectral theorem to T_G .

However, we will also need to know that T_G maps $L^2([a, b])$ into $C([a, b])$. This is true for any integral operator with continuous kernel:

Lemma 5.4.2. *Assume $K : [a, b] \times [a, b] \rightarrow \mathbb{C}$ is continuous. Then the associated integral operator $T_K : L^2([a, b]) \rightarrow L^2([a, b])$ maps $L^2([a, b])$ into $C([a, b])$.*

Proof. Let $f \in L^2([a, b])$ and let $\varepsilon > 0$. Note that the Cauchy-Schwarz inequality gives that

$$M := \int_{[a, b]} |f| \, dm \leq \left(\int_{[a, b]} 1 \, dm \right)^{1/2} \left(\int_{[a, b]} |f|^2 \, dm \right)^{1/2} = \sqrt{b-a} \|f\|_2 < \infty,$$

where m denotes the Lebesgue measure on $[a, b]$. As the continuous function K is automatically uniformly continuous on the compact set $R := [a, b] \times [a, b]$, we can find $\delta > 0$ such that

$$|K(x_1, t_1) - K(x_2, t_2)| < \varepsilon/M$$

whenever $(x_1, t_1), (x_2, t_2) \in R$ and $|x_2 - x_1| < \delta, |t_2 - t_1| < \delta$.

Let now $x_0 \in [a, b]$. Then for every $t \in [a, b]$ and all $x \in [a, b]$ such that $|x - x_0| < \delta$, we have

$$|K(x, t) - K(x_0, t)| < \varepsilon/M.$$

5. An introduction to Sturm-Liouville theory

Thus we get

$$\begin{aligned} \left| [T_K(f)](x) - [T_K(f)](x_0) \right| &= \left| \int_{[a,b]} (K(x,t) - K(x_0,t)) f(t) dm(t) \right| \\ &\leq \int_{[a,b]} |K(x,t) - K(x_0,t)| |f(t)| dm(t) \\ &\leq \varepsilon/M \int_{[a,b]} |f| dm = \varepsilon \end{aligned}$$

for all $x \in [a, b]$ such that $|x - x_0| < \delta$. This shows that $T_K(f)$ is continuous at x_0 .

Since x_0 was an arbitrary point of $[a, b]$, $T_K(f) \in C([a, b])$. ■

Theorem 5.4.3. *Assume that D is 1-1, and consider the Sturm-Liouville problem*

$$D(y) = \lambda y \quad \text{with } y \in Y.$$

Then the following assertions hold:

- *The eigenvalues for this problem form a countable set $\{\lambda_k : k \in \mathbb{N}\}$ of non-zero distinct real numbers satisfying that $|\lambda_k| \rightarrow \infty$ as $k \rightarrow \infty$.*
- *For each $k \in \mathbb{N}$ the eigenspace $E_{\lambda_k} = \{y \in Y : D(y) = \lambda_k y\}$ is one-dimensional.*
- *If y_k is a unit vector in E_{λ_k} for each $k \in \mathbb{N}$, then $\{y_k : k \in \mathbb{N}\}$ is an orthonormal basis for $L^2([a, b])$.*

Proof. We first observe that 0 is not an eigenvalue of T_G .

Indeed, since Y is dense in $H := L^2([a, b])$, and $Y = T_G(C([a, b]))$, we have

$$H = \overline{Y} \subseteq \overline{T_G(H)} \subseteq H,$$

hence $\overline{T_G(H)} = H$. Thus, we get $\ker(T_G)^\perp = \overline{T_G(H)} = H$, i.e., $\ker(T_G) = \{0\}$.

Applying the spectral theorem to T_G , we obtain that the eigenvalues of T_G form a countable set $\{\mu_k : k \in \mathbb{N}\}$ of non-zero distinct real numbers satisfying that $\mu_k \rightarrow 0$ as $k \rightarrow \infty$. (This set can not be finite: otherwise, T_G would have finite rank, which not possible since $\overline{T_G(H)} = H$.)

Let $k \in \mathbb{N}$, and set $\lambda_k = \mu_k^{-1} \neq 0$. Let $f_k \in H$ be an eigenfunction for T_G associated to μ_k . Since $T_G(f_k) = \mu_k f_k$, we get

$$f_k = \lambda_k T_G(f_k). \tag{5.4.1}$$

5.4. A special case

As Lemma 5.4.2 gives that $T_G(f_k) \in C([a, b])$, this gives that $f_k \in C([a, b])$. Hence, $T_G(T_G(f_k)) \in Y$. But (5.4.1) implies that

$$f_k = \lambda_k^2 T_G(T_G(f_k)),$$

so we get that $f_k \in Y$. Now, applying D to (5.4.1), we get

$$D(f_k) = \lambda_k f_k.$$

This shows that λ_k is an eigenvalue of D , and f_k is an eigenfunction for D associated to λ_k .

Now, Proposition 5.3.3 tells us that $E_{\lambda_k} := \{y \in Y : D(y) = \lambda_k y\}$ is one-dimensional. Hence, we have $E_{\lambda_k} = \text{Span}\{f_k\}$. Further, one readily checks that E_{λ_k} is also the eigenspace of T_G associated to μ_k .

We note that D can not have other eigenvalues than the λ_k 's (for if D had one such eigenvalue, then T_G would have an eigenvalue different from all μ_k 's, which is not the case). Further, we note that

$$\lim_{k \rightarrow \infty} |\lambda_k| = \lim_{k \rightarrow \infty} |\mu_k|^{-1} = \infty$$

since $\lim_{k \rightarrow \infty} \mu_k = 0$.

Finally, if we set $y_k := \pm(\|f_k\|_2)^{-1} f_k \in E_{\lambda_k}$ for each $k \in \mathbb{N}$, then we also get from the spectral theorem for T_G that $\{y_k : k \in \mathbb{N}\}$ is an orthonormal basis for H consisting of eigenfunctions for D . ■

Example 5.4.4. To illustrate this theorem, let us consider the Sturm-Liouville system

$$-y'' = \lambda y \text{ on } [0, \pi], \text{ with } Y = \{y \in C^2([0, \pi]) : y(0) = y(\pi) = 0\}.$$

Thus we have $D(y) = -y''$, $y \in Y$.

Let $\lambda \in \mathbb{C}$. We consider first the case $\lambda = 0$. It is straightforward to check that the only function y in Y satisfying $-y'' = 0$ is $y = 0$. Thus, 0 is not an eigenvalue of D , i.e., D is 1-1, so Theorem 5.4.3 applies in this case. We can determine the eigenvalues and the eigenfunctions of D explicitly as follows.

Assume $\lambda \neq 0$, and write $\lambda^{1/2} = r + is$ with $(r, s) \in \mathbb{R}^2 \setminus \{(0, 0)\}$.³ A basis for $S_\lambda := \{y \in C^2([0, \pi]) : -y'' = \lambda y\}$ is given by

$$\{e^{i\lambda^{1/2}x}, e^{-i\lambda^{1/2}x}\} = \{e^{-sx}(\cos(rx) + i\sin(rx)), e^{sx}(\cos(rx) - i\sin(rx))\}.$$

³We could here have used that we know that all the possible eigenvalues of D are real, so that we need only to consider $\lambda \in \mathbb{R} \setminus \{0\}$. However this would not shorten our discussion significantly.

5. An introduction to Sturm-Liouville theory

If $y \in S_\lambda$, say $y(x) = C_1 e^{-sx}(\cos(rx) + i \sin(rx)) + C_2 e^{sx}(\cos(rx) - i \sin(rx))$, then $y \in Y$ if and only if

$$\begin{cases} C_1 + C_2 = 0, \\ C_1 e^{-s\pi}(\cos(r\pi) + i \sin(r\pi)) + C_2 e^{s\pi}(\cos(r\pi) - i \sin(r\pi)) = 0. \end{cases}$$

This gives that $E_\lambda = S_\lambda \cap Y$ is non-trivial if and only if

$$e^{-s\pi}(\cos(r\pi) + i \sin(r\pi)) = e^{s\pi}(\cos(r\pi) - i \sin(r\pi)),$$

and it is elementary to deduce that this happens if and only if $s = 0$ and $r = k$ for some $k \in \mathbb{Z} \setminus \{0\}$, in which case $\lambda = k^2 \in \mathbb{N}$ and $E_\lambda = \text{Span}\{\sin(kx)\}$.

This means that the distinct eigenvalues of this Sturm-Liouville system are $\lambda_k = k^2$, $k \in \mathbb{N}$, with associated normalized eigenfunctions $y_k(x) = \sqrt{2/\pi} \sin(kx)$. This is in accordance with Theorem 5.4.3. Note that it follows from this theorem that $\{\sqrt{2/\pi} \sin(kx) : k \in \mathbb{N}\}$ is an orthonormal basis for $L^2([0, \pi])$.

For completeness, we also compute the Green's function G and $T_G(f)$ for $f \in C([0, \pi])$. One computes easily that

$$L_0 = \{y \in C^2([0, \pi]) : y'' = 0, y(0) = 0\} = \text{Span}\{u\},$$

where $u(x) = x$, while

$$R_0 = \{y \in C^2([0, \pi]) : y'' = 0, y(\pi) = 0\} = \text{Span}\{v\},$$

where $v(x) = x - \pi$. Thus we get that

$$W = W_{u,v}(x) = u(x)v'(x) - v(x)u'(x) = x - (x - \pi) = \pi$$

for all $x \in [0, \pi]$. Moreover, the Green's function $G : [0, \pi] \times [0, \pi] \rightarrow \mathbb{R}$ is then given by

$$G(x, t) = \frac{1}{\pi} \cdot \begin{cases} x(\pi - t) & \text{if } 0 \leq x \leq t \leq \pi, \\ t(\pi - x) & \text{if } 0 \leq t \leq x \leq \pi. \end{cases}$$

Moreover,

$$[T_G(f)](x) = \int_0^\pi G(x, t) f(t) dt = \frac{1}{\pi} \left((\pi - x) \int_0^x t f(t) dt + x \int_x^\pi (\pi - t) f(t) dt \right)$$

for all $f \in C([0, \pi])$ and $x \in [0, \pi]$.

Note that determining the eigenvalues of T_G by direct computation is not an easy task. Anyhow, you should verify that $\sin(kx)$ is an eigenfunction for T_G associated with the eigenvalue $\mu_k = k^{-2}$ for each $k \in \mathbb{N}$. $\square$

5.5 The general case

In this final section, we consider the general case, i.e., we don't assume that D is 1-1. The idea now is to show that there exists some $\mu \in \mathbb{R}$ which is not an eigenvalue of D , and consider the operator $D_\mu : Y \rightarrow C([a, b])$ defined by $D_\mu(y) = -y'' + qy - \mu y$. Then 0 will not be an eigenvalue of D_μ (otherwise there would be some $y \in Y \setminus \{0\}$ such that $D_\mu(y) = 0$, i.e., $D(y) = \mu y$, and μ would be an eigenvalue of D , giving a contradiction). Hence, we will be able to apply Theorem 5.4.3 to the Sturm-Liouville system $D_\mu(y) = \lambda' y$ with $y \in Y$, and deduce some interesting consequences for our original Sturm-Liouville problem.

Since we know that the possible eigenvalues of D are all real numbers, we may think: why not just pick some $\mu \in \mathbb{C} \setminus \mathbb{R}$? The problem with such a choice is that the function $q_\mu(x) := q(x) - \mu$ for $x \in [a, b]$ will not be real-valued, hence that the Sturm-Liouville system associated with D_μ will not match our requirements.

Lemma 5.5.1. *There exists some $\mu \in \mathbb{R}$ which is not an eigenvalue of D .*

Proof. We have seen earlier that $L^2([a, b])$ has a countable orthonormal basis, say $\{u_k\}_{k \in \mathbb{N}}$. We will show that this implies that D has countably many distinct eigenvalues. Since $\mathbb{R}$ is uncountable, the assertion to be proven will clearly follow.

Assume (for contradiction) that D has uncountably many distinct eigenvalues. Then we can pick a unit vector in each of the associated eigenspaces. As all eigenspaces of D are orthogonal to each other, this means that there exists an orthonormal subset Γ of Y which is uncountable.

Let $k \in \mathbb{N}$. Then it follows from Bessel's inequality that

$$M_k := \sup_{A \subseteq \Gamma, A \text{ finite}} \left\{ \sum_{\gamma \in A} |\langle u_k, \gamma \rangle|^2 \right\} \leq \|u_k\|_2^2 = 1 < \infty.$$

This implies that the set

$$U_{k,n} := \left\{ \gamma \in \Gamma : |\langle u_k, \gamma \rangle| \geq \frac{1}{n} \right\}$$

is finite for every $n \in \mathbb{N}$. Indeed, if $U_{k,n}$ was infinite for some n , then we could find an infinite sequence $\{\gamma_m\}_{m \in \mathbb{N}}$ of distinct elements in $U_{k,n}$, and this would give that

$$M_k \geq \sup_{m \in \mathbb{N}} \left\{ \sum_{\ell=1}^m |\langle u_k, \gamma_\ell \rangle|^2 \right\} \geq \sup_{m \in \mathbb{N}} \left\{ m \cdot \frac{1}{n^2} \right\} = \infty,$$

contradicting that $M_k < \infty$. Setting now $U_k := \{\gamma \in \Gamma : \langle u_k, \gamma \rangle \neq 0\}$, we get that

$$U_k = \bigcup_{n \in \mathbb{N}} U_{k,n}$$

5. An introduction to Sturm-Liouville theory

is countable (being a countable union of finite sets). Hence, the countable union $U := \bigcup_{k \in \mathbb{N}} U_k$ is a countable subset of Γ . As Γ is uncountable, there must exist some $\gamma \in \Gamma \setminus U$. But then $\gamma \notin U_k$ for every $k \in \mathbb{N}$, so we have

$$\langle u_k, \gamma \rangle = 0 \quad \text{for every } k \in \mathbb{N}.$$

This says that γ is orthogonal to every u_k , so we must have $\gamma = 0$ (since $\{u_k\}_{k \in \mathbb{N}}$ is an orthonormal basis for $L^2([a, b])$). This contradicts the fact that every element of Γ is a unit vector. We can therefore conclude that D has countably many distinct eigenvalues. $\blacksquare$

We can now state our main result about regular Sturm-Liouville systems:

Theorem 5.5.2. *Consider the Sturm-Liouville problem*

$$D(y) = \lambda y \quad \text{with } y \in Y.$$

Then the following assertions hold:

- *The eigenvalues for this problem form a countable set $\{\lambda_k : k \in \mathbb{N}\}$ of distinct real numbers satisfying that $|\lambda_k| \rightarrow \infty$ as $k \rightarrow \infty$.*
- *For each $k \in \mathbb{N}$ the eigenspace $E_{\lambda_k} = \{y \in Y : D(y) = \lambda_k y\}$ is one-dimensional.*
- *If y_k is a unit vector in E_{λ_k} for each $k \in \mathbb{N}$, then $\{y_k : k \in \mathbb{N}\}$ is an orthonormal basis for $L^2([a, b])$.*

Proof. By Lemma 5.5.1 we can find some $\mu \in \mathbb{R}$ which is not an eigenvalue of D . We define $D_\mu : Y \rightarrow C([a, b])$ by $D_\mu(y) = -y'' + qy - \mu y$ and consider the Sturm-Liouville system $D_\mu(y) = \lambda' y$ on Y . Then, as 0 is not an eigenvalue of D_μ , we may apply Theorem 5.4.3 to D_μ . This gives:

- The eigenvalues of D_μ form a countable set $\{\lambda'_k : k \in \mathbb{N}\}$ of non-zero distinct real numbers satisfying that $|\lambda'_k| \rightarrow \infty$ as $k \rightarrow \infty$.
- For each $k \in \mathbb{N}$ the eigenspace $E'_{\lambda'_k} = \{y \in Y : D_\mu(y) = \lambda'_k y\}$ is one-dimensional.
- If v_k is a unit vector in $E'_{\lambda'_k}$ for each $k \in \mathbb{N}$, then $\{v_k : k \in \mathbb{N}\}$ is an orthonormal basis for $L^2([a, b])$.

Now, for $y \in Y$, we obviously have $D(y) = \lambda y$ if and only if $D_\mu(y) = (\lambda - \mu)y$. This implies that the set consisting of all eigenvalues of D is the countable set of distinct real numbers given by $\{\lambda_k : k \in \mathbb{N}\}$, where $\lambda_k := \lambda'_k + \mu$ for each $k \in \mathbb{N}$.

Moreover, the eigenspace E_{λ_k} of D associated to each λ_k is then equal to $E'_{\lambda'_k}$, which is one-dimensional.

Finally, if y_k is a unit vector in E_{λ_k} for each $k \in \mathbb{N}$, then $y_k = \pm v_k$ for every $k \in \mathbb{N}$, so the last assertion clearly follows. ■

5.6 Exercises

Exercise 5.1. Find the eigenvalues and eigenfunctions of the Sturm-Liouville system $-y'' = \lambda y$ on the given interval with the following boundary conditions:

- a) $[a, b] = [0, \pi]$, $y'(0) = 0$, $y'(\pi) = 0$.
- b) $[a, b] = [0, \pi]$, $y'(0) = 0$, $y(\pi) = 0$.
- c) $[a, b] = [0, 2\pi]$, $y(0) = 0$, $y(2\pi) = 0$.
- d) $[a, b] = [0, 1]$, $y(0) = 0$, $y(1) + y'(1) = 0$.

Exercise 5.2. Consider a Sturm-Liouville system $D(y) = \lambda y$ on Y as in (5.1.4), with boundary conditions as in (5.1.2).

Assume that the following two extra conditions holds:

- a) $q(x) \geq 0$ for all $x \in [a, b]$,
- b) $\alpha_1 \alpha_2 \leq 0$ and $\beta_1 \beta_2 \geq 0$.

Show that the eigenvalues of D are all non-negative.

Exercise 5.3. Consider a Sturm-Liouville system $D(y) = \lambda y$ as in (5.1.4), but where

$$Y = \{y \in C^2([a, b]) \mid y(a) \text{ or } y'(a) = 0; y(b) \text{ or } y'(b) = 0\}.$$

Show that the distinct eigenvalues of D may ordered so that

$$\lambda_1 < \lambda_2 < \lambda_3 < \cdots \text{ and } \lim_{k \rightarrow \infty} \lambda_k = \infty.$$

Exercise 5.4. Assume D is 1-1, as in section 4. Show that the distinct eigenvalues of D satisfy

$$\sum_{k=1}^{\infty} \frac{1}{|\lambda_k|^2} < \infty.$$

Exercise 5.5. Show that the space

$$Y = \{y \in C^2([a, b]) : y \text{ satisfies the conditions (5.1.2)}\}$$

is dense in $L^2([a, b])$.

