

Example: The integral (Fredholm) operator.

Banach space version: $X = C[a, b]$ with $\|f\|_\infty = \max\{|f(t)| : t \in [a, b]\}$.

For $K: [a, b] \times [a, b] \rightarrow \mathbb{C}$ continuous, define $T_K: X \rightarrow X$ by

$$(T_K f)(s) = \int_a^b K(s, t) f(t) dt.$$

Let $M = \|K\| = \sup\{|K(s, t)| : (s, t) \in [a, b] \times [a, b]\} > 0$. ^{assump}

Then $T_K \in B(X)$ with $\|T_K\| \leq M \cdot (b-a) \cdot M$.

(Hint: use uniform continuity of K to show that $T_K f \in X$ for $f \in X$, thus T_K is well defined. Then use properties of the integral to show $\|T_K f\|_\infty \leq M \cdot (b-a) \cdot \|f\|_\infty, \forall f \in X$.)

Fact: $T \in K(X)$. - Need: the Arzela-Ascoli theorem.

Hilbert space versions: $\mathcal{H} = L^2[a, b]$, e.g. $L^2[0, 1]$.

$$\langle f, g \rangle = \int_{[a, b]} f(s) \overline{g(s)} ds, f, g \in \mathcal{H}.$$

(i) $K: [a, b] \times [a, b] \rightarrow \mathbb{C}$ continuous. $\rightarrow$ Then $\|K\|_\infty \leq M < \infty$.

(ii) $K \in L^2([a, b] \times [a, b])$. Then $\|K\|_2 < \infty$.

In either case, $(T_K f)(s) = \int_{[a, b]} K(s, t) f(t) dt, f \in \mathcal{H}$.

As before, need $T_K f \in L^2[a, b]$ for every $f \in L^2[a, b]$. (Hint: first for $f \in C[a, b]$, then use density of $C[a, b]$ in $\mathcal{H}$ in $\|\cdot\|_2$ norm.)

Show $\|T_K f\|_2 \leq \|K\|_2 \cdot \|f\|_2, \forall f \in \mathcal{H}$ (thus, T_K bounded)

Fact: $T_K \in K(\mathcal{H})$. Need: the Arzela-Ascoli theorem.

Example Multiplication operators.

(2)

For $1 \leq p < \infty$, $X = \ell^p(\mathbb{N}) = \{ \{x_n\}_n \mid x_n \in \mathbb{K}, \forall n, \sum_n |x_n|^p < \infty \}$

Let $\lambda = \{\lambda_n\} \in \ell^\infty(\mathbb{N})$. Define $M_\lambda: \ell^p(\mathbb{N}) \rightarrow \ell^p(\mathbb{N})$ by
 $(M_\lambda x)_n = \lambda_n x_n, n \in \mathbb{N}$.

Show: $\|M_\lambda\| = \|\lambda\|_\infty$. (Exercise)

Claim: $M_\lambda \in \mathcal{K}(\ell^p(\mathbb{N}))$ whenever $\lim_{n \rightarrow \infty} \lambda_n = 0$.

Enough to show: $M_\lambda \in \overline{\mathcal{F}(\ell^p(\mathbb{N}))}$.

Define $M_\lambda^{(k)}: \ell^p(\mathbb{N}) \rightarrow \ell^p(\mathbb{N})$, by $(x_1, x_2, \dots) \mapsto (x_1, x_2, \dots, x_k, 0, 0, \dots)$

$$M_\lambda^{(k)}: (x_1, x_2, \dots) \mapsto (\lambda_1 x_1, \dots, \lambda_k x_k, 0, 0, \dots)$$

Then $M_\lambda^{(k)} \in \mathcal{B}(X)$ for every k . Moreover, $M_\lambda^{(k)}$ has finite rank $\leq k$.

Note: $\lim_{k \rightarrow \infty} M_\lambda^{(k)} = M_\lambda$ in $\mathcal{B}(X)$. For example, by seeing

that with $\lambda^{(k)} = \{\lambda_n^{(k)}\}$, $\lambda_n^{(k)} = \begin{cases} \lambda_n, & 1 \leq n \leq k \\ 0, & n > k \end{cases}$, we have $\lambda^{(k)} \in \ell^\infty(\mathbb{N})$, with $\lim_{k \rightarrow \infty} \|\lambda^{(k)} - \lambda\|_\infty = 0$, and

$$M_\lambda^{(k)} = M_{\lambda^{(k)}} \in \mathcal{F}(X).$$

Observations about compact operators and finite dimensionality

1) If X fin. dim, then $\mathcal{K}(X) = \mathcal{B}(X)$ (indeed, $\mathcal{B}(X) = \mathcal{F}(X) = \mathcal{K}(X)$).

2) If X inf. dim, then $\mathcal{K}(X) \neq \mathcal{B}(X)$.

Easiest way to see this: $I_X \in \mathcal{B}(X)$, $I: X \rightarrow X$, $Ix = x$, $x \in X$.

But $I_X(B_X) = B_X$ not compact, closed and bounded.

3) If $T \in \mathcal{K}(X, Y)$, then $T(X)$ and $\overline{T(X)}$ are separable.

Hint: Let $B_r = \{x \in X \mid \|x\| < r\}$, $r > 0$. Then $\overline{T(B_r)}$ compact, so separable. Cover $T(X)$ by $T(B_r)$.

Compact operators on Hilbert spaces

Theorem [B, Theorem 4.2.1]

Let X be a normed space and $\mathcal{H}$ a Hilbert space (over K). Then

$$\overline{\mathcal{K}(X, \mathcal{H})} = \mathcal{K}(X, \mathcal{H})$$

In particular, $\overline{\mathcal{K}(\mathcal{H})} = \mathcal{K}(\mathcal{H})$

Proof. Know " $\subseteq$ ". Need $\mathcal{K}(X, \mathcal{H}) \subseteq \overline{\mathcal{K}(X, \mathcal{H})}$. We give a different proof than [B, Thm 4.2.1].

Let $T \in \mathcal{K}(X, \mathcal{H})$. If T has finite rank, the result is clear.

Assume $T(X)$ is infinite-dimensional (in $\mathcal{H}$).

By earlier observation, $\overline{T(X)}$ is separable, so it is a separable Hilbert space. Pick $\{e_n\}_{n \in \mathbb{N}} \subseteq \overline{T(X)}$ an o.n.b.

For every $k \geq 1$, let P_k = the orthogonal projection onto $\text{span}\{e_1, \dots, e_k\}$.

Set $T_k := P_k T$ for $k \geq 1$. Then $T_k \in \mathcal{K}(X, \mathcal{H})$, $\forall k \geq 1$. finite dimensional
(so closed!)

We show: $\lim_{k \rightarrow \infty} \|T_k - T\| = 0$.

Assume this is not the case. There exists $\varepsilon > 0$ s.t. $\|T_k - T\| \geq \varepsilon$ for all $k \geq n$. So, we can find $\{x_k\}_k \in X$ with $\|x_k\| \leq 1$ and:

$$(*) \quad \|(T_k - T)(x_k)\| \geq \varepsilon \text{ for all } k \geq 1.$$

Since T is compact, then by passing to a sequence and renaming, we may assume that $\{T(x_k)\}_k$ converges to some $y \in \overline{T(X)}$.

$$\text{Then } \|(T_k - T)(x_k)\| = \|(P_k - I)Tx_k\| \neq 0$$

$$\leq \|(P_k - I)(Tx_k - y)\| + \|(P_k - I)y\|$$

$$\leq \|Tx_k - y\| \xrightarrow{\text{as } k \rightarrow \infty} 0 \quad \quad \quad \sum_{n=k+1}^{\infty} |y_n|^2 \rightarrow 0 \text{ as } k \rightarrow \infty$$

This contradicts (*). In all, $\lim_{k \rightarrow \infty} \|T_k - T\| = 0$ so $T \in \overline{\mathcal{K}(X, \mathcal{H})}$.

Corollary Let $\mathcal{H}$ be a Hilbert space. We have

17/09/2025
(4)

$$K(\mathcal{H}) = \overline{F(\mathcal{H})}.$$

Corollary Let $T \in K(\mathcal{H})$. Then $T^* \in K(\mathcal{H})$.

Proof: First, show that $S \in F(\mathcal{H}) \iff S^* \in F(\mathcal{H})$.

(Exercise. Hint: Solution(1): Write $\mathcal{H} = \ker S^* \oplus (\ker S^*)^\perp$.

Solution(2): Write $S = \sum_{j=1}^m S_{w_j, y_j}$ for $w_j, y_j \in \mathcal{H}$. Find S^* .

Now, if $T \in K(\mathcal{H})$, then $T = \lim_{k \rightarrow \infty} T_k$, $T_k \in F(\mathcal{H})$. Then $T_k^* \in F(\mathcal{H})$ by first part,

Now use that $\lim_{k \rightarrow \infty} T_k^* = T^*$, so $T^* \in F(\mathcal{H})$.

See exercise in [R, chapter 3]

Hilbert-Schmidt operators

Suppose $\mathcal{H}$ is a separable inf. dimensional Hilbert space.

Note: Any o.n.b. must be countable (we already used this!)

If there existed an uncountable o.n.b. B , then no sequence can be dense in $\mathcal{H}$, contradicting separability of $\mathcal{H}$.

Why? Note that $\|u - u'\| = (\|u\|^2 + \|u'\|^2)^{1/2} = \sqrt{2}$. Use the open sets $\{x \in \mathcal{H} \mid \|x - u\| < \frac{1}{2}\}$ for all $u \in B$.

Definition Let $\mathcal{H}$ be separable, inf. dim. We say $T \in B(\mathcal{H})$ is a Hilbert-Schmidt operator if

$$\sum_{j=1}^{\infty} \|Tu_j\|^2 < \infty, \quad (1)$$

where $\{u_j\}_{j=1}^{\infty}$ is some orthonormal basis for $\mathcal{H}$.

Define the Hilbert-Schmidt norm: $\|T\|_2 = \left(\sum_{j=1}^{\infty} \|Tu_j\|^2 \right)^{1/2}$.

Need: (1) does not depend on choice of o.n.b. (2) is a norm.

Lemma If $\{u_j\}_{j=1}^{\infty}, \{v_j\}_{j=1}^{\infty}$ are o.n. bases in a sep-Hilbert space $\mathcal{H}$ and $T \in B(\mathcal{H})$, then

$$\sum_{j=1}^{\infty} \|Tu_j\|^2 = \sum_{j=1}^{\infty} \|T^*v_j\|^2.$$

Proof: $\sum_{j=1}^{\infty} \|Tu_j\|^2 = \sum_{j=1}^{\infty} \langle Tu_j, Tu_j \rangle$ $Tu_j = \sum_{k=1}^{\infty} \langle Tu_j, v_k \rangle v_k$

$$= \sum_{j=1}^{\infty} \left\langle \sum_{k=1}^{\infty} \langle Tu_j, v_k \rangle v_k, Tu_j \right\rangle$$

$$= \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \langle Tu_j, v_k \rangle \langle v_k, Tu_j \rangle$$

$$= \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \langle u_j, T^*v_k \rangle \langle T^*v_k, u_j \rangle$$

$$= \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \langle u_j, T^*v_k \rangle \langle T^*v_k, u_j \rangle$$

$$= \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \overline{\langle T^*v_k, u_j \rangle} \langle T^*v_k, u_j \rangle$$

$$= \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \langle T^*v_k, \langle T^*v_k, u_j \rangle u_j \rangle$$

$$= \sum_{k=1}^{\infty} \langle T^*v_k, T^*v_k \rangle$$

$$= \sum_{k=1}^{\infty} \|T^*v_k\|^2.$$

$$\langle Tu_j, u_j \rangle \langle u_j, Tu_j \rangle =$$

$$= \langle u_j, T^*u_j \rangle \langle T^*u_j, u_j \rangle$$

Similarly, $\sum_{j=1}^{\infty} \|T^*u_j\|^2 = \sum_{k=1}^{\infty} \|T^*v_k\|^2$, which implies the lemma $\square$.

Define $\mathcal{HS}(\mathcal{H}) = \{T \in B(\mathcal{H}) \mid T \text{ is Hilbert-Schmidt}\}$

Prop: $\mathcal{F}(\mathcal{H}) \subseteq \mathcal{HS}(\mathcal{H}) \subseteq \mathcal{K}(\mathcal{H})$ ($\mathcal{H} \neq \{0\}$ separable)
subspace.