

Proposition [B, Prop 4.2.8]. Let $\mathcal{H} \neq \{0\}$ separable. We have:

(1) $\mathcal{F}(\mathcal{H}) \subseteq \mathcal{HS}(\mathcal{H}) \subseteq \mathcal{K}(\mathcal{H})$.

(2) $\mathcal{HS}(\mathcal{H})$ is a subspace of $\mathcal{K}(\mathcal{H})$. Further $\|\cdot\|_2$ is a norm on $\mathcal{HS}(\mathcal{H})$ and $\|T\| \leq \|T\|_2$. Also, $T \in \mathcal{HS}(\mathcal{H}) \Rightarrow T^* \in \mathcal{HS}(\mathcal{H})$

Proof: Assume $\mathcal{H}$ inf. dim. By a result in Lecture 10, if $\{u_j\}_j$ is an o.n.b., $\sum_{j=1}^{\infty} \|Tu_j\|^2 < \infty \Leftrightarrow \sum_{j=1}^{\infty} \|T^*u_j\|^2 < \infty$.

So $T \in \mathcal{HS}(\mathcal{H}) \Rightarrow T^* \in \mathcal{HS}(\mathcal{H})$

Let $T_1, T_2 \in \mathcal{HS}(\mathcal{H})$, let $\{u_j\}_j$ o.n.b. Then $a := \{\|T_1 u_j\|\}_j \in \ell^2(\mathbb{N})$.

$\forall j: \|(T_1 + T_2)u_j\|^2 \leq (\|T_1 u_j\| + \|T_2 u_j\|)^2$ $b := \{\|T_2 u_j\|\}_j \in \ell^2(\mathbb{N})$

$\sum_{j=1}^{\infty} \|(T_1 + T_2)u_j\|^2 \leq \sum_{j=1}^{\infty} (\|T_1 u_j\| + \|T_2 u_j\|)^2 = \|a + b\|_2^2$

For $\|T\| \leq \|T\|_2$, choose $x \neq 0$, let $u_1 = \frac{x}{\|x\|}$, complete to an o.n.b. $\{u_j\}_j$

$\|Tx\| = \|x\| \cdot \|Tu_1\| \leq \|x\| \cdot \|T\|_2 \Rightarrow \|T\| \leq \|T\|_2$

Thus $T_1 + T_2 \in \mathcal{HS}(\mathcal{H})$. In fact, $\|T_1 + T_2\|_2 \leq \|T_1\|_2 + \|T_2\|_2$, showing the triangle inequality and that $\mathcal{HS}(\mathcal{H})$ is a subspace of $\mathcal{B}(\mathcal{H})$. (Clearly $\lambda \in \mathbb{C}, T \in \mathcal{HS}(\mathcal{H}) \Rightarrow \lambda T \in \mathcal{HS}(\mathcal{H})$).

Have: $\|T\|_2 = 0 \Rightarrow (\sum_j \|Tu_j\|^2)^{1/2} = 0 \Rightarrow Tu_j = 0, \forall j \Rightarrow Tx = 0, \forall x \in \mathcal{H}$, so $T = 0$. We get $\|\cdot\|_2: \mathcal{HS}(\mathcal{H}) \rightarrow [0, \infty)$ is a norm.

For $T \in \mathcal{HS}(\mathcal{H})$, let $\{u_j\}_{j=1}^{\infty}$ o.n.b. for $\mathcal{H}$, let $P_k = P_{\text{span}\{e_1, \dots, e_k\}}$ for every $k \geq 1$.

Then $TP_k \in \mathcal{K}(\mathcal{H})$, and $TP_k \in \mathcal{HS}(\mathcal{H})$ as $\sum_{j=1}^{\infty} \|TP_k u_j\|^2 = \sum_{j=1}^k \|Tu_j\|^2 < \infty$.

$\Rightarrow \|T - TP_k\| \stackrel{\text{show}}{\leq} \|T - TP_k\|_2 \leq (\sum_{k+1}^{\infty} \|Tu_j\|^2)^{1/2} \rightarrow 0$ as $k \rightarrow \infty$. $\square$

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(2)

Finish the proof of:

Proposition: If $T \in \mathcal{K}(\mathcal{H})$, then $\text{Im}(T)$ and $\overline{\text{Im}(T)}$ are separable.

As in hint: Write $T(\mathcal{H}) = \bigcup_{r \in \mathbb{Q}, r > 0} T(B_r)$ for $B_r = \{x \in \mathcal{H} \mid \|x\| < r\}$.

Have $T(B_r)$ compact, thus $\overline{T(B_r)}$ and $T(B_r)$ separable.

Corollary (of result saying $\mathcal{F}(\mathcal{H}) = \mathcal{K}(\mathcal{H})$ when $\mathcal{H}$ is a Hilbert space)
 $T \in \mathcal{K}(\mathcal{H}) \Rightarrow T^* \in \mathcal{K}(\mathcal{H})$

Proof: Let $S \in \mathcal{F}(\mathcal{H})$. Since $S^* \in \mathcal{B}(\mathcal{H})$ and $\ker S^*$ is closed, we write $\mathcal{H} = \ker S^* \oplus (\ker S^*)^\perp$. So $\forall x \in \mathcal{H}$, $x = x_1 + x_2$, $x_1 \in \ker S^*$, $x_2 \in (\ker S^*)^\perp$. Then $S^*x = S^*x_1 + S^*x_2 = 0 + S^*x_2 = S^*x_2$.

But $(\ker S^*)^\perp = (S(\mathcal{H}))^\perp = \overline{S(\mathcal{H})} = S(\mathcal{H})$, since $S(\mathcal{H})$ is closed. So $S^*x_2 \in S^*(S(\mathcal{H}))$. Since $S(\mathcal{H})$ is finite dimensional, we get $S^* \in \mathcal{F}(\mathcal{H})$.

Note that by interchanging S with S^* shows:

$$S \in \mathcal{F}(\mathcal{H}) \Leftrightarrow S^* \in \mathcal{F}(\mathcal{H})$$

To finish the proof of the corollary, use that $\overline{\mathcal{F}(\mathcal{H})} = \mathcal{K}(\mathcal{H})$. $\square$

Alternatively; see [R, Lemma 15.7].

Use that $TT^* \in \mathcal{K}(\mathcal{H})$ by proposition in Lecture 9. For a bounded sequence $\{x_n\}_n \in \mathcal{H}$, show that $\{T^*x_n\}_n$ is Cauchy, thus convergent.

Why?

$$\begin{aligned} \|T^*(x_n - x_m)\|^2 &= |\langle T^*(x_n - x_m), T^*(x_n - x_m) \rangle| \\ &= |\langle TT^*(x_n - x_m), x_n - x_m \rangle| \\ &\leq \underbrace{\|TT^*x_n - TT^*x_m\|}_{\leq 2\|x_n\|_\infty} \|x_n - x_m\| < \varepsilon \end{aligned}$$

for all $n \geq m \geq N_\varepsilon$.

This is convergent
 after passing to a subsequence,
 and relabelling.

$\square$

X Banach space $\mathcal{T}$ over $\mathbb{C}$

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③

Definition Given $T \in B(X)$, a scalar $\lambda \in \mathbb{C}$ is an eigenvalue of T if there exists $x \neq 0$ in X s.t. $Tx = \lambda x$.

Let $\sigma_p(T) = \{\lambda \in \mathbb{C} \mid \lambda \text{ eigenvalue of } T\}$.

Note: $\sigma_p(T) = \{\lambda \in \mathbb{C} \mid (T - \lambda I)x = 0 \text{ for some } x \in X \setminus \{0\}\}$.

—: if $Tx = \lambda x$ for $x \neq 0$, then $T(\frac{x}{\|x\|}) = \lambda \cdot \frac{x}{\|x\|}$. Thus, if $\lambda \in \sigma_p(T)$, we can always pick $u \in X, \|u\| = 1$ s.t. $(T - \lambda I)u = 0$.

For $\lambda \in \sigma_p(T)$, the set $\ker(T - \lambda I) = \{x \in X \mid Tx = \lambda x\}$ is a subspace of X , called the eigenspace corresponding to λ .

Define also $\sigma(T) = \{\lambda \in \mathbb{C} \mid T - \lambda I \text{ is not invertible in } B(X)\}$.

Theorem. Let X be infinite dimensional. If $T \in K(X)$, then $0 \in \sigma(T)$.

Proof: Let $T \in K(X)$. Assume $0 \notin \sigma(T)$. Then T is invertible in $B(X)$. Let $T^{-1} \in B(X)$ be its inverse. We get: $I = TT^{-1} \in K(X)$ by earlier proposition, which is false (the unit ball in X is not compact). $\square$

Lemma ① Let $T \in K(H)$, H Hilbert space, let $\lambda \in \mathbb{C} \setminus \{0\}$.

Then $\ker(T - \lambda I)$ is finite dimensional.

By the Gram-Schmidt procedure,

Proof Assume $\ker(T - \lambda I)$ is infinite dimensional. We can find an orthonormal sequence $\{x_n\}_{n \in \mathbb{N}}$ in $\ker(T - \lambda I)$. Thus

$Tx_n = \lambda x_n$ and $\langle x_n, x_m \rangle = \delta_{n,m}$ for all $n, m \in \mathbb{N}$.

Then $\|Tx_n - Tx_m\|^2 = \|\lambda x_n - \lambda x_m\|^2 = |\lambda|^2 (\|x_n\|^2 + \|x_m\|^2) = 2|\lambda|^2 > 0$, so $\{Tx_n\}_n$ admits no convergent subsequences, contradicting $T \in K(H)$. $\square$

For $\lambda \in \sigma_p(T)$, denote $E_\lambda = \ker(T - \lambda I)$.

Warning: For $T \in B(X)$, even in $K(X)$, $\sigma_p(T)$ can be empty.

So $\sigma_p(T) \neq \sigma(T)$ in general.

Lemma 2. Let $T \in \mathcal{K}(\mathcal{H})$ with $T = T^*$. ($\mathcal{H}$ Hilbert space).

(i) If $\lambda \in \sigma_p(T)$, then $\lambda \in \mathbb{R}$.

(ii) If $\lambda, \lambda' \in \sigma_p(T)$ with $\lambda \neq \lambda'$, then $E_\lambda \perp E_{\lambda'}$, meaning $\langle x, y \rangle = 0$ for every $x \in E_\lambda, y \in E_{\lambda'}$.

Proof: (i) Suppose $\lambda \in \sigma_p(T)$. Then $\langle Tx, x \rangle = \langle \lambda x, x \rangle = \lambda \|x\|^2 \in \mathbb{R}$ for any $x \in E_\lambda$ with $\|x\| = 1$. Since $\langle Tx, x \rangle \in \mathbb{R}$, we get $\lambda \in \mathbb{R}$.

(ii) Let $x \in E_\lambda, y \in E_{\lambda'}$ for $\lambda \neq \lambda'$ in $\sigma_p(T)$.

$$\langle Tx, y \rangle = \langle \lambda x, y \rangle = \lambda \langle x, y \rangle$$

$$\langle x, Ty \rangle = \langle x, \lambda' y \rangle \stackrel{(i)}{=} \lambda' \langle x, y \rangle, \text{ so } \underbrace{(\lambda - \lambda')}_{\neq 0} \langle x, y \rangle = 0. \text{ Thus } \langle x, y \rangle = 0. \quad \square$$

Key result ([R, Thm. 16.3], [B, Lemma 4.3.3]) Let $T \in \mathcal{K}(\mathcal{H})$ be self-adjoint. Then T has an eigenvalue $\lambda \in \mathbb{R}$ satisfying $|\lambda| = \|T\|$.

Proof: The result is trivially true if $T = 0$.

Assume $T \neq 0$. By the result about numerical range N_T , we can find $\{x_n\}_{n \in \mathbb{N}}$ in $\mathcal{H}$ s.t. $\|x_n\| = 1, \forall n$ and $\lim_{n \rightarrow \infty} |\langle Tx_n, x_n \rangle| = \|T\|$.

Since $\langle Tx, x \rangle \in \mathbb{R}$ for all x , we may assume (by passing to a subsequence and relabelling): $\lim_{n \rightarrow \infty} \langle Tx_n, x_n \rangle = \lambda$, where $\lambda = \|T\|$ or $\lambda = -\|T\|$.

By again passing to a subsequence and relabelling, assume $\{Tx_n\}_n$ converges to $y \in \mathcal{H}$. Then $0 < |\lambda| \leq \lim_n \|Tx_n\| \cdot \underbrace{\|x_n\|}_{=1} = \|y\|$, so $y \neq 0$.

$$\begin{aligned} \text{Now } \|Tx_n - \lambda x_n\|^2 &= \langle Tx_n - \lambda x_n, Tx_n - \lambda x_n \rangle = \|Tx_n\|^2 - 2\lambda \langle Tx_n, x_n \rangle + \lambda^2 \|x_n\|^2 \\ &\leq \|T\|^2 - 2\lambda \langle Tx_n, x_n \rangle + \lambda^2 = 2\lambda(\lambda - \langle Tx_n, x_n \rangle) \rightarrow 0 \text{ as } n \rightarrow \infty. \end{aligned}$$

Thus $y = \lim_{n \rightarrow \infty} Tx_n = \lambda \lim_{n \rightarrow \infty} x_n$. It follows (by continuity of T) that

$$Ty = T(\lambda \lim_{n \rightarrow \infty} x_n) = \lambda \lim_{n \rightarrow \infty} Tx_n = \lambda y, \text{ so } y \in E_\lambda, \text{ as needed. (because } y \neq 0). \quad \square$$

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We want to prove the Spectral theorem for $T \in \mathcal{K}(\mathcal{H})$, self-adjoint. ⑤

In big lines, this says: $\exists$ o.n.b. for $\overline{T(\mathcal{H})}$ ^{even for $\mathcal{H}$} consisting of eigenvectors for T !

First: Corollary of Key result ([R, Proposition 16.4], [B, Thm 4.3.4 (a), (b), (c)]).

Let $T \in \mathcal{K}(\mathcal{H})$ with $T \neq 0$ and $T = T^*$. Then $\sigma_p(T) \subseteq \mathbb{R}$ is nonempty.

Moreover, $\sigma_p(T)$ is either finite or consists of a sequence that converges to 0. For $\lambda \in \sigma_p(T)$, $\lambda \neq 0$, the eigenspace $E_\lambda = \ker(T - \lambda I)$ is fin. dim.

Observation: If $m_\lambda = \dim(E_\lambda)$, we call m_λ the (geometric) multiplicity of λ .

Proof: By the key result, either $\|T\| \in \sigma_p(T)$ or $-\|T\| \in \sigma_p(T)$.

Thus, $\sigma_p(T) \neq \emptyset$. By Lemma ②, $\sigma_p(T) \subseteq \mathbb{R}$. In fact, $\sigma_p(T) \subseteq [-\|T\|, \|T\|]$.

[Claim]: For every $\varepsilon > 0$, the set $L_\varepsilon = \{\lambda \in \sigma_p(T) \mid |\lambda| \geq \varepsilon\}$ is finite (or $\emptyset$).

Proof of claim: Suppose a contradiction that L_ε is infinite.

We can then pick a sequence $\{\lambda_n\}_{n \in \mathbb{N}} \subseteq L_\varepsilon$ with $\lambda_n \neq \lambda_m$ whenever $n \neq m$. Choose $x_n \in E_{\lambda_n}$ for each $n \geq 1$. By Lemma ②,

$\langle x_n, x_m \rangle = 0$ for $n \neq m$. By normalizing (ie. $\frac{x_n}{\|x_n\|}$), may assume $\|x_n\| = 1$. Thus $\{x_n\}_n$ is a bounded sequence and $\forall n, m$,

$$\|Tx_n - Tx_m\|^2 = \|\lambda_n x_n - \lambda_m x_m\|^2 = \|\lambda_n x_n\|^2 + \|\lambda_m x_m\|^2$$

$= |\lambda_n|^2 + |\lambda_m|^2 \geq 2\varepsilon^2$. Then $\{Tx_n\}_n$ cannot have any convergent seq., a contradiction.

Then $L = \{\lambda \in \sigma_p(T) \mid \lambda \neq 0\} = \bigcup_{n \in \mathbb{N}} L_{1/n}$, at most countable.

Suppose L is countable, say $L = \{\lambda_k\}_{k \in \mathbb{N}}$. Let $\varepsilon > 0$. Since

L_ε is finite, $\exists k_\varepsilon \in \mathbb{N}$ s.t. $|\lambda_k| < \varepsilon$ for all $k \geq k_\varepsilon$. So $\lim_{k \rightarrow \infty} \lambda_k = 0$.

That $\dim(E_\lambda) < \infty$ was shown in Lemma ①.

Obs: The claim is true for arbitrary $T \in \mathcal{K}(\mathcal{H})$. (Not necessarily self-adjoint!)