

Let $a < b$ real, let $\textcircled{1}$

Integral operator (Banach space) $X = (C[a, b], \|\cdot\|_\infty)$

For $K: [a, b] \times [a, b] \rightarrow \mathbb{C}$ continuous, define $T_K: X \rightarrow X$, $(T_K f)(s) = \int_a^b K(s, t) f(t) dt$

Let $M = \sup \{ |K(s, t)| : (s, t) \in [a, b] \times [a, b] \} > 0$ (assume).

(I) Claim: For $f \in C[a, b]$, $T_K f \in C[a, b]$.

Proof: Need $s \mapsto (T_K f)(s)$ continuous for $f \in C[a, b]$.

First, note that for fixed (but arbitrary) $s \in [a, b]$, the function

$t \mapsto K(s, t) f(t)$ is continuous, as a product of continuous functions on $[a, b]$.

Given $s_0 \in [a, b]$, given $\varepsilon > 0$, we must find $\delta > 0$ s.t.

$$(1) |s - s_0| < \delta \Rightarrow |T_K(f)(s) - T_K(f)(s_0)| < \varepsilon.$$

$$|T_K(f)(s) - T_K(f)(s_0)| = \left| \int_a^b (K(s, t) - K(s_0, t)) f(t) dt \right|$$

$$\leq \int_a^b |K(s, t) - K(s_0, t)| \cdot |f(t)| dt$$

$$\leq \left(\int_a^b |K(s, t) - K(s_0, t)| dt \right) \cdot \|f\|_\infty.$$

By uniform continuity of K , find $\delta > 0$ so this can be made $< \frac{\varepsilon}{(b-a) \cdot \|f\|_\infty}$

We get (1) and (I) is proved.

whenever $|s - s_0| < \delta$.

(II) Claim: $T_K \in B(X)$ with $\|T_K\| \stackrel{(2)}{\leq} M \cdot (b-a)$.

Let $f \in X$. Then $\|T_K f\|_\infty = \sup_{s \in [a, b]} |(T_K f)(s)|$. But

$$|(T_K f)(s)| \leq \int_a^b |K(s, t)| \cdot |f(t)| dt \leq \left(\int_a^b |K(s, t)| dt \right) \cdot \|f\|_\infty \leq M(b-a) \cdot \|f\|_\infty.$$

Take supremum over s to get (II).

Integral operator (Banach space version) - continued.(III) Claim: $T \in \mathcal{K}(X)$.

Let $\{f_n\}_n \subseteq X$ be a bounded sequence. Thus there exists $C > 0$ s.t. $\|f_n\|_\infty < C$ for all $n \geq 1$. Need to show that $\{T_K f_n\}_n$ converges for some subsequence $\{f_{n_k}\}_k$.

Consider the collection of functions $\{T_K f_n | n \geq 1\}$ in $C[a, b]$. ^{real-valued} Will show that this is equicontinuous and bounded.

Boundedness is easy: $\|T_K f_n\|_\infty \leq \|T_K\| \cdot \|f_n\|_\infty \stackrel{(II)(2)}{\leq} M \cdot (b-a) \cdot C, \forall n$.

Equicontinuity: Let $\varepsilon > 0$. Need to find $\delta > 0$ s.t.

(3) $|s - s'| < \delta \Rightarrow |(T_K f_n)(s) - (T_K f_n)(s')| < \varepsilon$ for all $n \geq 1$.

But $|(T_K f_n)(s) - (T_K f_n)(s')| \leq \int_a^b |K(s, t) - K(s', t)| \cdot |f_n(t)| dt \leq$

$$\leq \left(\int_a^b |K(s, t) - K(s', t)| dt \right) \cdot C < \varepsilon$$

$< \frac{\varepsilon}{C \cdot (b-a)}$ for $|s - s'| < \delta$ found by uniform continuity of K .

This shows (3). By the Arzela-Ascoli theorem,

the family $\{T_K f_n | n \in \mathbb{N}\}$ is precompact in $C(X, \mathbb{R})$.

Then $\{T_K f_n | n \in \mathbb{N}\}$ admits a convergent subsequence.

(See MAT 2400 or [R]).

In all, $T_K \in \mathcal{K}(X)$.

[B, Exercise 4.1 b, c, d] X, Y, Z normed spaces over K , $T \in B(X, Y)$, $S \in B(Y, Z)$

(b) If $T \in K(X, Y)$, then $ST \in K(X, Z)$.

Let $\{x_n\}_n \subset X$ bounded. By compactness of T , there is a subsequence $\{Tx_{n_k}\}_k$ which converges.

But S is continuous, so $\{ST(x_{n_k})\}_k$ converges.

(c) If $S \in K(Y, Z)$, then $ST \in K(X, Z)$.

Let $\{x_n\} \subset X$ bounded, let $C > 0$ s.t. $\|x_n\| \leq C$ for all $n \geq 1$.

Then $\|Tx_n\| \leq \|T\| \cdot \|x_n\| \leq \|T\| \cdot C, \forall n$, so $\{Tx_n\}_n \subset Y$ is bounded.

Compactness of $S \Rightarrow \exists$ subseq. $\{S(Tx_{n_k})\}_k$ which converges.

(d) $K(X) := K(X, X)$. Deduce that

$ST \in K(X)$ if $T \in B(X)$ and $S \in K(X)$

or if $T \in K(X)$ and $S \in B(X)$.

Immediate from (b) and (c).

[B, Exercise 4.2]. Let $X = \ell^p(\mathbb{N})$, $\lambda \in \ell^\infty(\mathbb{N})$, $M_\lambda \in B(X)$ be the multiplication operator. Show: If $M_\lambda \in K(X)$, then $\lambda \in c_0(\mathbb{N})$

Proof: Assume $\lambda \notin c_0(\mathbb{N})$. There exists $\varepsilon > 0$ s.t. $|\lambda_n| \geq \varepsilon$ for infinitely many n 's. By passing to a subsequence, $|\lambda_{n_k}| \geq \varepsilon$ for all $k \geq 1$. Now let $\{e^{n_k}\}_{k \geq 1}$, $e_j^{n_k} = \begin{cases} 1, & j = n_k \\ 0, & \text{otherwise} \end{cases}$

So $\|e^{n_k}\|_p = 1, \forall k$.

$$\begin{aligned} \|M_\lambda e^{n_k} - M_\lambda e^{n_l}\|_p^p &= \sum_{j=1}^{\infty} |\lambda_j e_j^{n_k} - \lambda_j e_j^{n_l}|^p \\ &= |\lambda_{n_k}|^p + |\lambda_{n_l}|^p > 2\varepsilon^p. \end{aligned}$$

Thus, $\{M_\lambda e^{n_k}\}$ has no convergent subsequence, a contradiction.

[B, Exercise 4.3] X normed space, $\mathcal{H}$ Hilbert space, $T \in \mathcal{K}(X, \mathcal{H})$.

Show that $\overline{T(X)}$ is separable.

Proved in class.

[B, Exercise 4.5]. Let $\mathcal{H}$ be a Hilbert space and $P \in B(\mathcal{H})$ with $P^2 = P$. Show $P \in \mathcal{F}(\mathcal{H}) \Leftrightarrow P$ is compact.

Clearly $P \in \mathcal{F}(\mathcal{H})$ is compact.

Assume P compact. Will show $P \in \mathcal{F}(\mathcal{H})$. Assume $\dim P(\mathcal{H}) = \infty$, and choose an o.n. sequence $\{x_n\} \in P(\mathcal{H})$.

Then $x_n^2 = P y_n, \forall n$.

But $P x_n = P(P y_n) = P^2 y_n = P y_n = x_n$, so for $n, m \geq 1$,

$$\|P x_n - P x_m\|^2 = \|x_n - x_m\|^2 = \|x_n\|^2 + \|x_m\|^2 = 2, \text{ thus}$$

$\{P x_n\}_n$ cannot have any conv. sequence, a contradiction.

So $\dim P(\mathcal{H}) < \infty$.

After the case $X = C[a, b]$.

The integral operator on $H = L^2([a, b])$ with Lebesgue measure and $K: [a, b] \times [a, b] \rightarrow \mathbb{C}$ cont.

Need: $T_K f \in L^2[a, b]$ for any $f \in L^2[a, b]$.

As

$$\|T_K f\|_2^2 = \int_a^b |(T_K f)(s)|^2 ds = \int_a^b \left| \int_a^b \underbrace{K(s, t)}_{\langle K(s, \cdot), f \rangle} f(t) dt \right|^2 ds$$

$$\leq \int_a^b \|K(s, \cdot)\|_2^2 \cdot \|f\|_2^2 ds$$

$$= \left(\int_a^b \int_a^b |K(s, t)|^2 dt ds \right) \cdot \|f\|_2^2 \text{ so } \|T_K\| \leq \|K\|_2.$$

Either show directly: $\forall \{f_n\}_n \subset L^2(a, b)$, f_n cont, $\{K f_n\}_n$ equicontinuous.
Or, show $T_K \in \mathcal{HS}(\mathcal{H})$.

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To show that $T_K \in \mathcal{K}(\mathcal{H})$, $\mathcal{H} = L^2[a, b]$.

Take $\{f_n\}_n \in C[a, b]$ bounded, say $\|f_n\|_2 \leq M_1$.

Let $g_n = T_K f_n \in C[a, b]$. Need to find a subsequence such that $\{g_n\}_n$ converges in $\|\cdot\|_2$ -norm. Claim $\{g_n\}_n$ is equicontinuous.

First, $\{g_n\}_n$ is bounded:

$$\begin{aligned} \text{For every } s, |g_n(s)| &= |(T_K f_n)(s)| \leq \int_a^b |K(s, t) f_n(t)| dt \\ &\leq \int_a^b \underbrace{|K(s, t)|}_{\leq M} \cdot |f_n(t)| dt \stackrel{\text{C-S}}{\leq} M \cdot \|1\|_2 \cdot \|f_n\|_2 \\ &\leq M \cdot M_1 \cdot \|1\|_2. \end{aligned}$$

constant function on $[a, b]$

$\Rightarrow \{g_n\}_n$ is uniformly bounded.

$\{g_n\}$ is equicontinuous: Let $\varepsilon > 0$. Need $\delta > 0$ s.t.

$$|s - s'| < \delta \Rightarrow |g_n(s) - g_n(s')| < \varepsilon.$$

$$\text{But } |g_n(s) - g_n(s')| \leq \int_a^b |K(s, t) - K(s', t)| \cdot |f_n(t)| dt$$

$$\text{Find } \delta > 0 \text{ so } \underbrace{|K(s, t) - K(s', t)|}_{< \frac{\varepsilon}{M \cdot \|1\|_2}} < \frac{\varepsilon}{M \cdot \|1\|_2} \text{ when } |s - s'| < \delta$$

$$< \frac{\varepsilon}{M \cdot \|1\|_2} \cdot \underbrace{\int_a^b |f_n(t)| dt}_{= \langle 1, f_n \rangle \leq \|1\|_2 \cdot \|f_n\|_2} < \varepsilon.$$

By Arzela-Ascoli, $\{g_n\}_n$ has a convergent subsequence.

More is true (1) $T_K \in \mathcal{HS}(\mathcal{H})$. See example 4.2.11 in Be'dos' notes.

(2) If $K(s, t) = \sum_{i=1}^m k_i(s) h_i(t)$ where $k_i, h_i \in C[a, b]$, then $T_K \in \mathcal{K}(\mathcal{H})$.

Example: $K(s, t) = \sin(s - t)$ on $[0, 1]$, example 4.1.5 in [B].